Uncountable collections of pairwise disjoint non-chainable tree-like continua in the plane

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March 19, 2011
STDC11

Definitions, Moore's Theorem

Continuum $\equiv$ compact connected metric space

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Theorem (R. L. Moore, 1928)

The plane $\mathbb{R}^2$ does not contain an uncountable collection of pairwise disjoint triods.

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If this is not all of them, then by (Jones, 1955) and (Hagopian, 1976), there must be another one which is hereditarily indecomposable and tree-like.

Indecomposable homogeneous plane continua and triods

Lemma (Hagopian, 1975)

Let M be an indecomposable homogeneous continuum in the plane. Then M does not contain a triod.

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- By homogeneity, each composant of M contains a copy of T
- This contradicts Moore's theorem □

Non-chainable tree-like continua in the plane

Theorem (Oversteegen & Tymchatyn, 1984)

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There exists an uncountable family of pairwise disjoint non-chainable tree-like continua in the plane.

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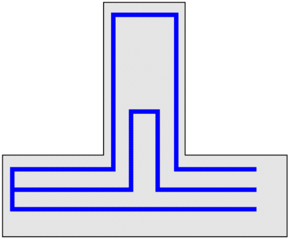

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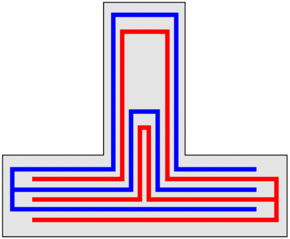

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Example (H, 2011)

Let X be the non-chainable continuum with span zero from (H, 2011).
Then $X \times \mathcal{C}$ is embeddable in the plane, where $\mathcal{C}$ is the middle-thirds Cantor set.
Moreover, if $p, q \in \mathcal{C}$ with $|p - q| < \varepsilon$, then there is a ε-homeomorphism of the plane to itself taking $X \times \{p\}$ to $X \times \{q\}$.

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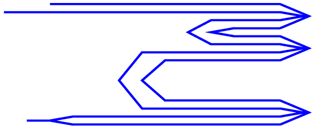

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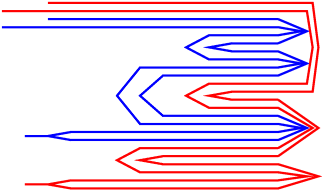

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1. Is there a *hereditarily indecomposable* non-chainable tree-like continuum X such that the plane contains an uncountable collection of pairwise disjoint copies of X?
2. If X is a tree-like continuum and $X \times \mathcal{C}$ embeds in the plane, must X have span zero?
3. Is there a hereditarily indecomposable non-chainable continuum with span zero?