Span Zero and Surjective Span Zero

L. C. Hoehn
(logan.hoehn@utoronto.ca)

University of Toronto

March 7, 2009
STDC09

Span

Let X be a continuum with metric d.

Let X be a continuum with metric d.

Definition (Lelek, 1964)

The *span of* X is the supremum of the numbers

$$\inf\{d(x,y) : (x,y) \in Z\}$$

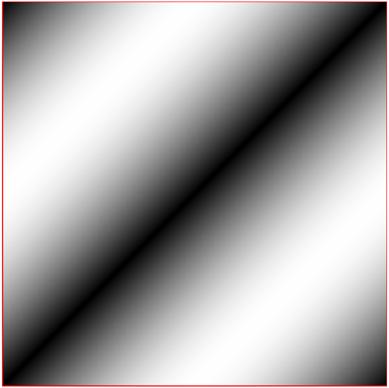
where Z ranges over all subcontinua of X^2 with $\pi_1(Z) = \pi_2(Z)$.

(here $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$.)

Span example: Unit Circle

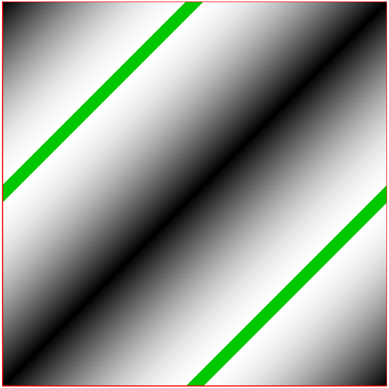
Consider the unit circle $\mathbb{S}^1$ in $\mathbb{R}^2$ with the Euclidean metric:

Consider the unit circle $\mathbb{S}^1$ in $\mathbb{R}^2$ with the Euclidean metric:



Span example: Unit Circle

Consider the unit circle $\mathbb{S}^1$ in $\mathbb{R}^2$ with the Euclidean metric:



Define Z as shown. This witnesses that the span of $\mathbb{S}^1$ is ≥ 2.

Span and Surjective Span

Definition (Lelek, 1964)

The *span of* X is the supremum of the numbers

$$\inf\{d(x,y) : (x,y) \in Z\}$$

where Z ranges over all subcontinua of X^2 with $\pi_1(Z) = \pi_2(Z)$.

Span and Surjective Span

Definition (Lelek, 1964)

The *span of* X is the supremum of the numbers

$$\inf\{d(x,y) : (x,y) \in Z\}$$

where Z ranges over all subcontinua of X^2 with $\pi_1(Z) = \pi_2(Z)$.

Definition (Lelek, 1977)

The *surjective span of* X is the supremum of the numbers

$$\inf\{d(x,y) : (x,y) \in Z\}$$

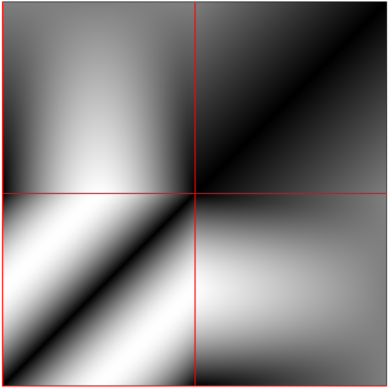
where Z ranges over all subcontinua of X^2 with $\pi_1(Z) = \pi_2(Z) = X$.

Span example: Noose Space 1

The noose space $N = \mathbb{S}^1 \cup [0, 1]$ in $\mathbb{R}^2$ with the Euclidean metric:

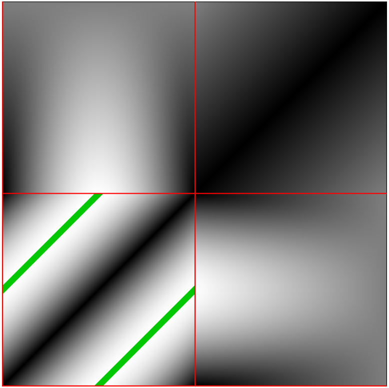
Span example: Noose Space 1

The noose space $N = \mathbb{S}^1 \cup [0,1]$ in $\mathbb{R}^2$ with the Euclidean metric:



Span example: Noose Space 1

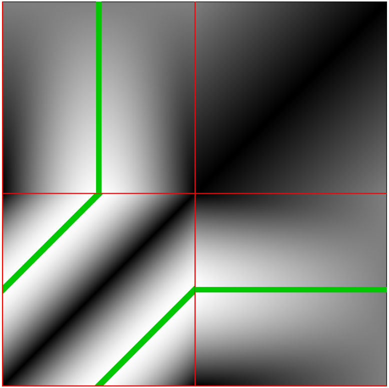
The noose space $N = \mathbb{S}^1 \cup [0,1]$ in $\mathbb{R}^2$ with the Euclidean metric:



This set Z_1 witnesses that the span of N is ≥ 2.

Span example: Noose Space 1

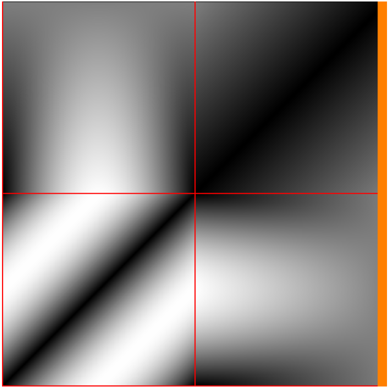
The noose space $N = \mathbb{S}^1 \cup [0,1]$ in $\mathbb{R}^2$ with the Euclidean metric:



This set Z_2 witnesses that the surjective span of N is ≥ 1.

Span example: Noose Space 1

The noose space $N = \mathbb{S}^1 \cup [0,1]$ in $\mathbb{R}^2$ with the Euclidean metric:



This set W shows that the surjective span of N is ≤ 1.

Span vs. Surjective Span

Question (Lelek, 1977)

Is it true that $\mathrm{Span}(X) \leq 2 \cdot \mathrm{SurjectiveSpan}(X)$ for every metric continuum X?

Span vs. Surjective Span

Question (Lelek, 1977)

Is it true that $\mathrm{Span}(X) \leq 2 \cdot \mathrm{SurjectiveSpan}(X)$ for every metric continuum X?

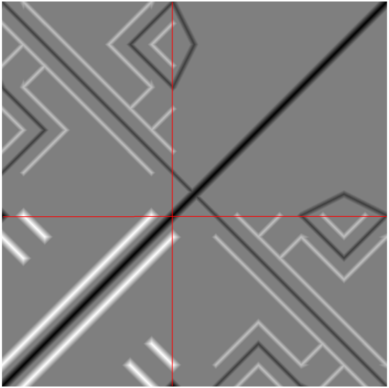
Answer: No. There is a metric on the noose space (shown on the next slide) so that the span is 1 and the surjective span is $\frac{1}{4}$.

Span example: Noose Space 2

The noose space $N = \mathbb{S}^1 \cup [0,1]$ in $\mathbb{R}^2$ with a different metric d:

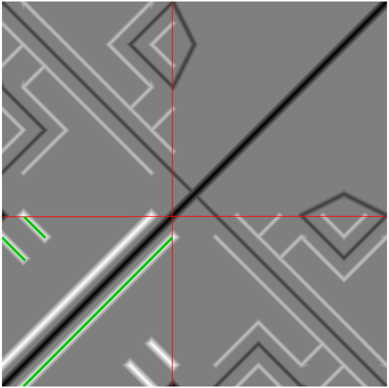
Span example: Noose Space 2

The noose space $N = \mathbb{S}^1 \cup [0, 1]$ in $\mathbb{R}^2$ with a different metric d:



Span example: Noose Space 2

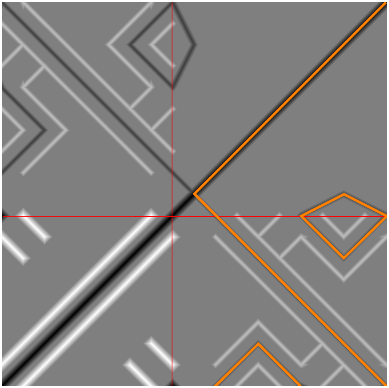
The noose space $N = \mathbb{S}^1 \cup [0,1]$ in $\mathbb{R}^2$ with a different metric d:



This set Z witnesses that the span of (N, d) is ≥ 1.

Span example: Noose Space 2

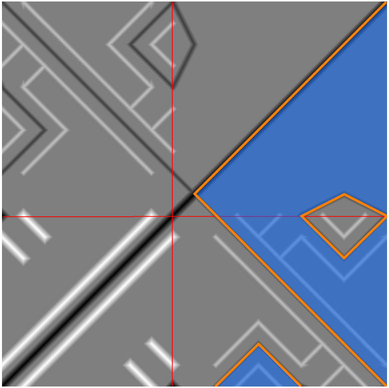
The noose space $N = \mathbb{S}^1 \cup [0,1]$ in $\mathbb{R}^2$ with a different metric d:



This set W shows that the surjective span of (N, d) is $\leq \frac{1}{4}$.

Span example: Noose Space 2

The noose space $N = \mathbb{S}^1 \cup [0,1]$ in $\mathbb{R}^2$ with a different metric d:



This set W shows that the surjective span of (N, d) is $\leq \frac{1}{4}$.

Span vs. Surjective Span

Question

Is there some $k \geq 4$ such that $\mathrm{Span}(X) \leq k \cdot \mathrm{SurjectiveSpan}(X)$ for every metric continuum X?

Span vs. Surjective Span

Question

Is there some $k \geq 4$ such that $\mathrm{Span}(X) \leq k \cdot \mathrm{SurjectiveSpan}(X)$ for every metric continuum X?

An affirmative answer to the above question would yield an affirmative answer to:

Question

Does a metric continuum have span zero if and only if it has surjective span zero?

Span Zero

Definition (Lelek, 1964)

The *span of* X is the supremum of the numbers

$$\inf\{d(x,y) : (x,y) \in Z\}$$

where Z ranges over all subcontinua of X^2 with $\pi_1(Z) = \pi_2(Z)$.

Span Zero

Definition (Lelek, 1964)

The *span of* X is the supremum of the numbers

$$\inf\{d(x,y) : (x,y) \in Z\}$$

where Z ranges over all subcontinua of X^2 with $\pi_1(Z) = \pi_2(Z)$.

Fact

A metric continuum X *has span zero iff* $Z \cap \Delta X \neq \emptyset$ *for every subcontinuum* Z *of* X^2 *with* $\pi_1(Z) = \pi_2(Z)$.

(here ΔX *denotes the diagonal* $\{(x,x) : x \in X\}$*.)*

Span Zero

Definition (Lelek, 1964)

The *span of* X is the supremum of the numbers

$$\inf\{d(x,y) : (x,y) \in Z\}$$

where Z ranges over all subcontinua of X^2 with $\pi_1(Z) = \pi_2(Z)$.

Fact

A metric continuum X *has span zero iff* $Z \cap \Delta X \neq \emptyset$ *for every subcontinuum* Z *of* X^2 *with* $\pi_1(Z) = \pi_2(Z)$.

(here ΔX *denotes the diagonal* $\{(x,x) : x \in X\}$*.)*

Span Zero

Definition

A continuum X has *span zero* if $Z \cap \Delta X \neq \emptyset$ for every subcontinuum Z of X^2 with $\pi_1(Z) = \pi_2(Z)$.

(here ΔX denotes the diagonal $\{(x, x) : x \in X\}$.)

Span Zero

Definition

A continuum X has *span zero* if $Z \cap \Delta X \neq \emptyset$ for every subcontinuum Z of X^2 with $\pi_1(Z) = \pi_2(Z)$.

(here ΔX denotes the diagonal $\{(x,x) : x \in X\}$.)

Definition

A continuum X has *surjective span zero* if $Z \cap \Delta X \neq \emptyset$ for every subcontinuum Z of X^2 with $\pi_1(Z) = \pi_2(Z) = X$.

Question (Lelek, 1971)

Does a metric continuum have span zero if and only if it is chainable?

Question (Lelek, 1971)

Does a metric continuum have span zero if and only if it is chainable?

There is a non-metric version of the definition of chainable:

Definition

A continuum X is *chainable* if every open cover for X has a chain refinement.
(a chain cover is a finite cover $\{U_1, \ldots, U_n\}$ such that $U_i \cap U_j \neq \emptyset$ iff $|i - j| \leq 1$.)

Span Zero and Chainability

Question (Lelek, 1971)

Does a metric continuum have span zero if and only if it is chainable?

There is a non-metric version of the definition of chainable:

Definition

A continuum X is *chainable* if every open cover for X has a chain refinement.
(a chain cover is a finite cover $\{U_1, \ldots, U_n\}$ such that $U_i \cap U_j \neq \emptyset$ iff $|i - j| \leq 1$.)

Fact

Any chainable continuum has span zero.

From Non-Metric to Metric

Van der Steeg (2003) describes a method for obtaining from a (non-metric) continuum X a metric continuum $\hat{X}$ and countable lattices L and K such that:

- L is a base for the closed sets of $\hat{X}$,
- K is a base for the closed sets of $\hat{X}^2$, and
- given any first order formula Φ in the language of set theory, Φ holds for L and K if and only if Φ holds for 2^X and 2^{X^2}.

From Non-Metric to Metric

Van der Steeg (2003) describes a method for obtaining from a (non-metric) continuum X a metric continuum $\hat{X}$ and countable lattices L and K such that:

- L is a base for the closed sets of $\hat{X}$,
- K is a base for the closed sets of $\hat{X}^2$, and
- given any first order formula Φ in the language of set theory, Φ holds for L and K if and only if Φ holds for 2^X and 2^{X^2}.

Using this third property, he showed:

From Non-Metric to Metric

Van der Steeg (2003) describes a method for obtaining from a (non-metric) continuum X a metric continuum $\hat{X}$ and countable lattices L and K such that:

- L is a base for the closed sets of $\hat{X}$,
- K is a base for the closed sets of $\hat{X}^2$, and
- given any first order formula Φ in the language of set theory, Φ holds for L and K if and only if Φ holds for 2^X and 2^{X^2}.

Using this third property, he showed:

$$\mathbf{X} \textbf{ not chainable} \quad \Rightarrow \quad \hat{\mathbf{X}} \textbf{ not chainable}$$

From Non-Metric to Metric

Van der Steeg (2003) describes a method for obtaining from a (non-metric) continuum X a metric continuum $\hat{X}$ and countable lattices L and K such that:

- L is a base for the closed sets of $\hat{X}$,
- K is a base for the closed sets of $\hat{X}^2$, and
- given any first order formula Φ in the language of set theory, Φ holds for L and K if and only if Φ holds for 2^X and 2^{X^2}.

Using this third property, he showed:

$$\begin{aligned} \textbf{X not chainable} \quad &\Rightarrow \quad \hat{\textbf{X}} \textbf{ not chainable} \\ \textbf{X has span non-zero} \quad &\Rightarrow \quad \hat{\textbf{X}} \textbf{ has span non-zero} \end{aligned}$$

From Non-Metric to Metric

Van der Steeg (2003) describes a method for obtaining from a (non-metric) continuum X a metric continuum $\hat{X}$ and countable lattices L and K such that:

- L is a base for the closed sets of $\hat{X}$,
- K is a base for the closed sets of $\hat{X}^2$, and
- given any first order formula Φ in the language of set theory, Φ holds for L and K if and only if Φ holds for 2^X and 2^{X^2}.

Using this third property, he showed:

$$\textbf{X not chainable} \Rightarrow \hat{\textbf{X}} \textbf{ not chainable}$$
$$\textbf{X has span non-zero} \Rightarrow \hat{\textbf{X}} \textbf{ has span non-zero}$$

It is also possible to show:

$$\textbf{X has surjective span zero} \Rightarrow \hat{\textbf{X}} \textbf{ has surjective span zero}$$

From Non-Metric to Metric

Van der Steeg (2003) describes a method for obtaining from a (non-metric) continuum X a metric continuum $\hat{X}$ and countable lattices L and K such that:

- L is a base for the closed sets of $\hat{X}$,
- K is a base for the closed sets of $\hat{X}^2$, and
- given any first order formula Φ in the language of set theory, Φ holds for L and K if and only if Φ holds for 2^X and 2^{X^2}.

Using this third property, he showed:

$$\mathbf{X \text{ not chainable}} \Rightarrow \mathbf{\hat{X} \text{ not chainable}}$$
$$\mathbf{X \text{ has span non-zero}} \Rightarrow \mathbf{\hat{X} \text{ has span non-zero}}$$

It is also possible to show:

$$\mathbf{X \text{ has surjective span zero}} \Rightarrow \mathbf{\hat{X} \text{ has surjective span zero}}$$

With extra machinery from model theory, Hart, van der Steeg, Bartošová (2008) show additionally that:

$$\mathbf{X \text{ has span zero}} \Rightarrow \mathbf{\hat{X} \text{ has span zero}}$$

Non-Metric Examples?

Upshot: We may as well look for non-metric continua which have span zero but are not chainable, or which have surjective span zero but not span zero.

Non-Metric Examples?

Upshot: We may as well look for non-metric continua which have span zero but are not chainable, or which have surjective span zero but not span zero.

Suggestions?

Non-Metric Examples?

Upshot: We may as well look for non-metric continua which have span zero but are not chainable, or which have surjective span zero but not span zero.

Suggestions?

Thank you!