Hereditarily equivalent continua in the plane

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Background & main result

Continuum $\equiv$ compact connected metric space

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Theorem (Oversteegen-H 2019)

The arc and pseudo-arc are the only hereditarily equivalent continua in $\mathbb{R}^2$.

ε-strips

Let $\varepsilon > 0$.

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Definition (Oversteegen-Tymchatyn 1982)

Let $f, g : [0,1] \to \mathbb{R}^2$ be piecewise linear such that

$$f([0,1]) \cap g([0,1]) = \emptyset \quad \text{and} \quad \|f(t) - g(t)\| < \varepsilon \quad \forall t.$$

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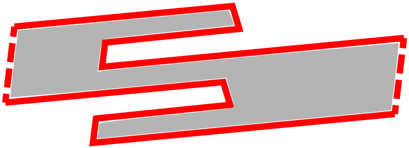

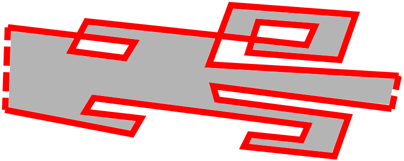

ε-strips, continued

Theorem (Oversteegen-Tymchatyn 1984)

If $X \subset \mathbb{R}^2$ is a hereditarily equivalent continuum, then there is a subcontinuum Y of X such that for all $\varepsilon > 0$, Y is contained in an ε-strip.

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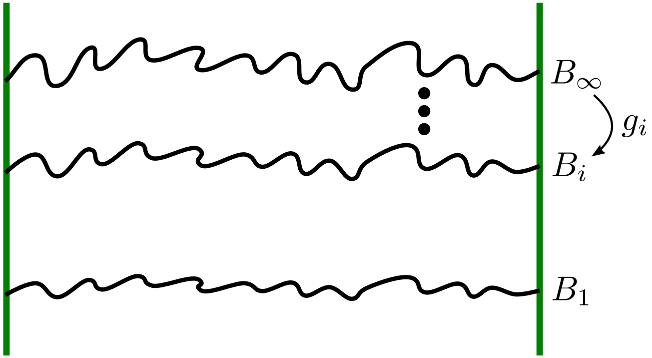

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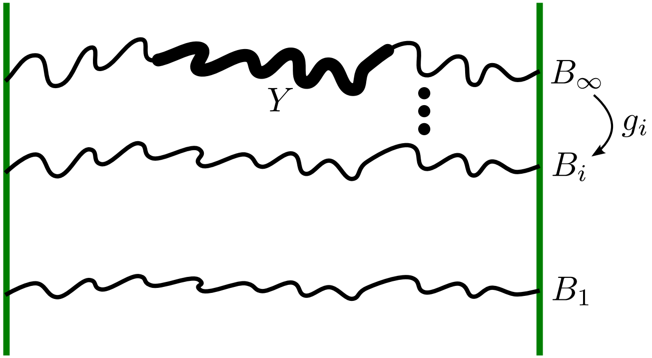

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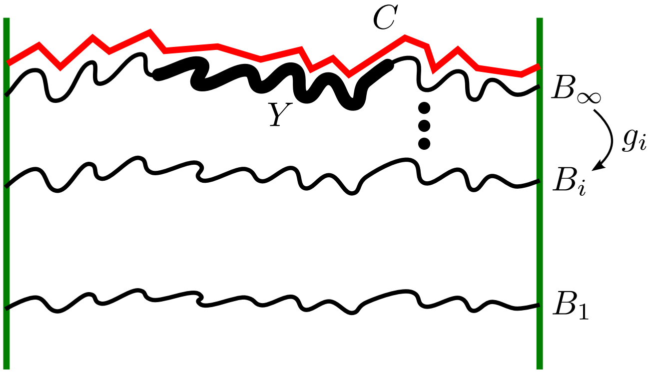

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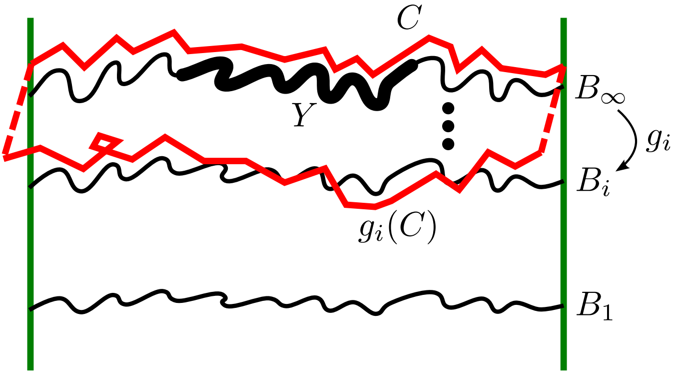

Separators in $G \times [0,1]$

Lemma

Let $G \subset \mathbb{R}^2$ be a graph contained in the ε-strip determined by f, g. Then

$$\{(x,t) \in G \times [0,1] : x \in \overline{f(t)g(t)}\}$$

separates $G \times \{0\}$ from $G \times \{1\}$ in $G \times [0,1]$.

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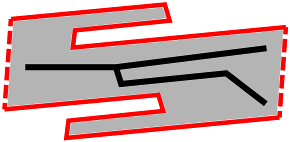

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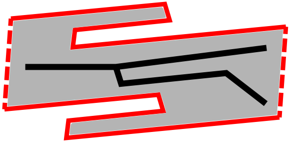

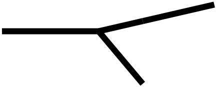

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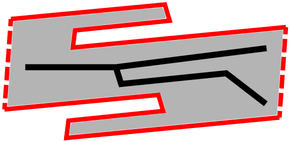

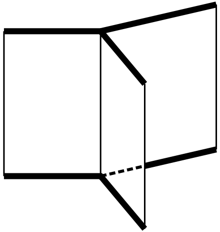

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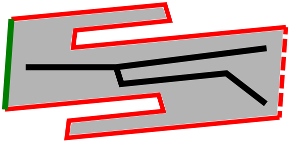

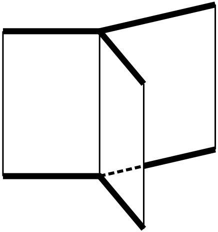

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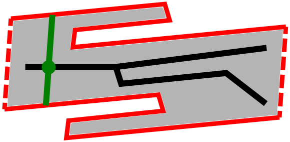

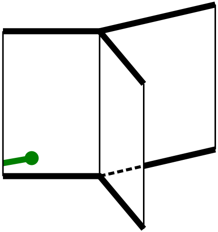

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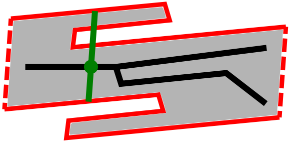

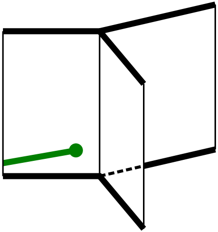

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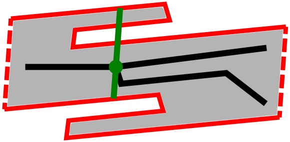

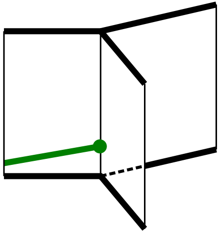

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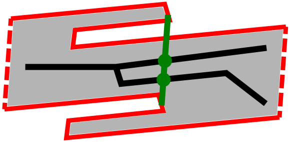

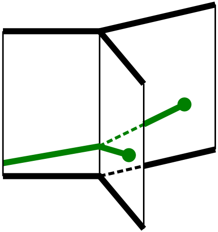

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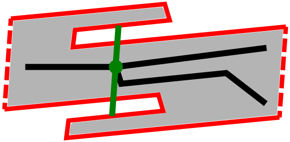

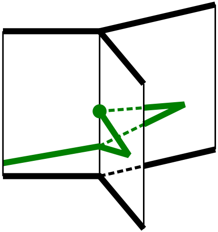

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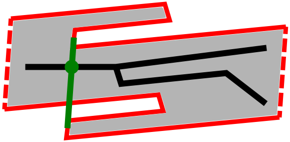

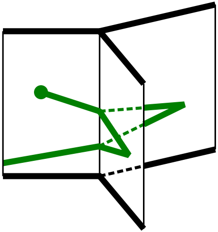

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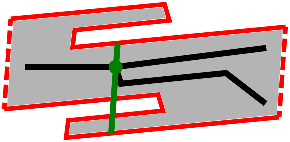

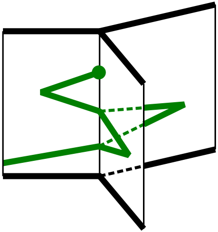

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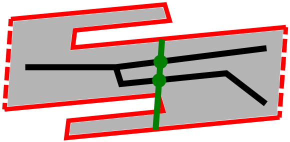

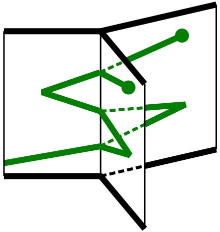

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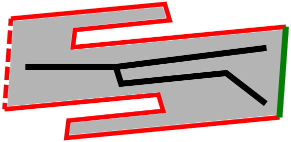

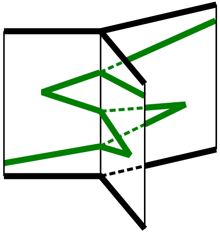

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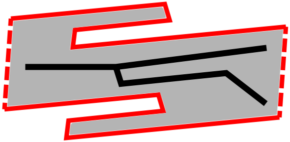
Theorem (Oversteegen-H 2016/2019)

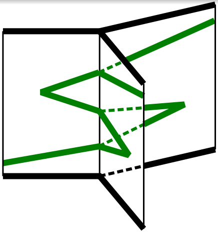
A continuum X is hereditarily indecomposable if and only if

$\forall$ *map $f : X \to G$ to a graph G*

$\forall$ *open $U \subset G \times (0,1)$ which separates $G \times \{0\}$ from $G \times \{1\}$ in $G \times [0,1]$*

$\exists$ $h : X \to U$ *with* $f = \pi_1 \circ h$.





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