Convex Metrics on Non-Compact Spaces

Ihor Stasyuk
(with Edward D.Tymchatyn)

Nipissing University

March 8, 2017

Introduction

Definition

Let (X, ρ) be a metric space. We say that ρ is *convex* if for each $x \neq y \in X$ there is an arc $A \subset X$ with end-points x and y such that $(A, \rho|_{A\times A})$ is isometric to the interval $[0, \rho(x, y)]$ in the real line.

Introduction

Definition

Let (X, ρ) be a metric space. We say that ρ is *convex* if for each $x \neq y \in X$ there is an arc $A \subset X$ with end-points x and y such that $(A, \rho|_{A \times A})$ is isometric to the interval $[0, \rho(x, y)]$ in the real line.

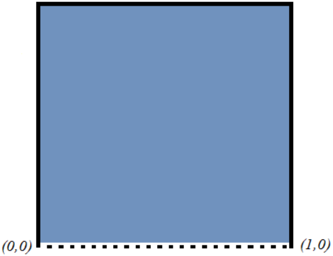
Theorem (Bing, 1949)

Every Peano continuum (i.e. compact, connected and locally connected metric space) admits a compatible convex metric.

- $\{\mathcal{U}_n\}_{n=1}^{\infty}$ - a decreasing sequence of finite partitions of X.
- $w_n(U)$ - the weight assigned to $U \in \mathcal{U}_n$. A subcollection of $\mathcal{U}_n$ has weight equal to the sum of the weights of its elements.
- An approximation to the distance $\rho(x, y)$ is the smallest of the weights $w_n(\mathcal{C}_n)$ of chains $\mathcal{C}_n$ in $\mathcal{U}_n$ from x to y.
- Then $\limsup \bigcup \mathcal{C}_n$ contains a line segment (with respect to ρ) from x to y.

Introduction

- Bing asked for an extension of his result to non-compact metric spaces. In the non-compact case, $\limsup \bigcup \mathcal{C}_n$ need not in general contain a connected subset from x to y.
- Consider for instance $X = ([0,1] \times [0,1]) \setminus ((0,1) \times \{0\})$ in its usual metric inherited from the plane and choose $x = (0,0)$ and $y = (1,0)$.



Property S

Definition

A metric space (X, ρ) has *property S* if for each $\varepsilon > 0$ there is a finite cover of X by connected sets of diameter less than ε.

- $\mathbb{R}$ in its usual metric does not have property S while $(0, 1)$ in its usual metric does. So this property is a metric property.
- if (X, d) has property S, then it is locally connected and totally bounded.
- every locally connected metric continuum has property S.

Convex metric on non-compact spaces

Theorem (J.Nikiel, M.Tuncali, E.D. Tymchatyn and S. (2013))

If X is a connected and locally arc-connected metric space with property S, then X admits a convex metric.

Partition of a set

Definition

A finite closed covering $\mathcal{V}$ of the metric space (X, d) is a *partition* of X if the following conditions are satisfied for all $U, V \in \mathcal{V}$:

- V and $\mathrm{int}(V)$ are connected and locally connected, and the first one of them is regular closed while the other is regular open,
- if $U \neq V$ then $U \cap V \subset \mathrm{bd}(U) \cap \mathrm{bd}(V)$.

If the mesh of $\mathcal{V}$ is less than $\varepsilon > 0$ then $\mathcal{V}$ is called an ε-partition of X.

Brick partitioning

Definition

We shall say that a partition $\mathcal{V}$ of X is a brick partition of X if the following conditions are satisfied for all $U, V \in \mathcal{V}$:

- $\mathrm{int}(V)$ is uniformly locally connected;
- $\mathrm{int}(U \cup V)$ is uniformly locally connected.

Theorem (Bing, 1949)

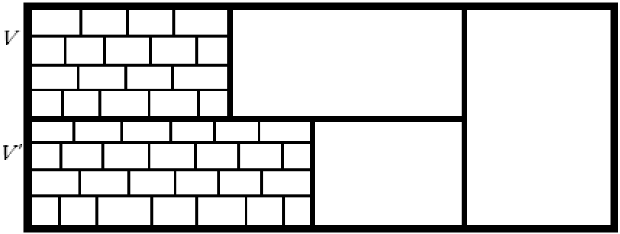
Each Peano continuum has a sequence $\{\mathcal{U}_i\}$ of brick partitionings such that $\mathcal{U}_i$ has mesh less than $1/i$ and $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$.

Core refinement

Definition

A collection $\mathcal{U}$ is a *core refinement* of a partition $\mathcal{V}$ of X if $\mathcal{U}$ is also a partition of X and the following conditions are satisfied for all $V, V' \in \mathcal{V}$:

- the union of all interior elements from $\mathcal{U}$ contained in V is connected,
- each boundary element from $\mathcal{U}$ meets an interior element from $\mathcal{U}$.
- if $V \cap V' \neq \varnothing$ then the union of interior elements from $\mathcal{U}$ that are contained in $V \cup V'$, is connected.

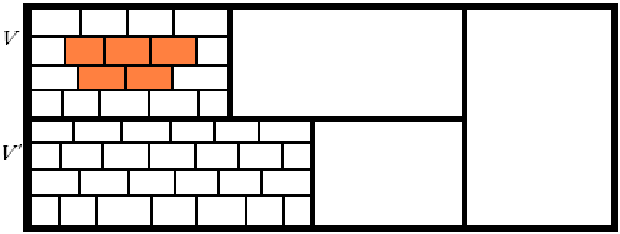


Core refinement

Definition

A collection $\mathcal{U}$ is a *core refinement* of a partition $\mathcal{V}$ of X if $\mathcal{U}$ is also a partition of X and the following conditions are satisfied for all $V, V' \in \mathcal{V}$:

- the union of all interior elements from $\mathcal{U}$ contained in V is connected,
- each boundary element from $\mathcal{U}$ meets an interior element from $\mathcal{U}$.
- if $V \cap V' \neq \varnothing$ then the union of interior elements from $\mathcal{U}$ that are contained in $V \cup V'$, is connected.

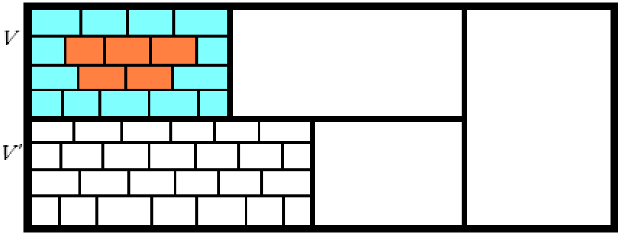


Core refinement

Definition

A collection $\mathcal{U}$ is a *core refinement* of a partition $\mathcal{V}$ of X if $\mathcal{U}$ is also a partition of X and the following conditions are satisfied for all $V, V' \in \mathcal{V}$:

- the union of all interior elements from $\mathcal{U}$ contained in V is connected,
- each boundary element from $\mathcal{U}$ meets an interior element from $\mathcal{U}$.
- if $V \cap V' \neq \varnothing$ then the union of interior elements from $\mathcal{U}$ that are contained in $V \cup V'$, is connected.

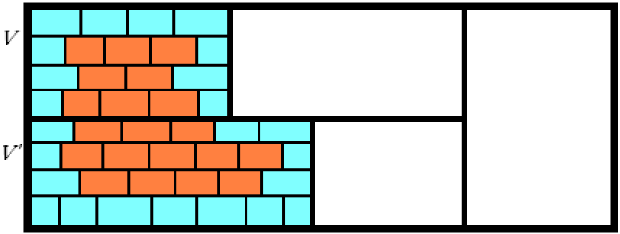


Core refinement

Definition

A collection $\mathcal{U}$ is a *core refinement* of a partition $\mathcal{V}$ of X if $\mathcal{U}$ is also a partition of X and the following conditions are satisfied for all $V, V' \in \mathcal{V}$:

- the union of all interior elements from $\mathcal{U}$ contained in V is connected,
- each boundary element from $\mathcal{U}$ meets an interior element from $\mathcal{U}$.
- if $V \cap V' \neq \varnothing$ then the union of interior elements from $\mathcal{U}$ that are contained in $V \cup V'$, is connected.



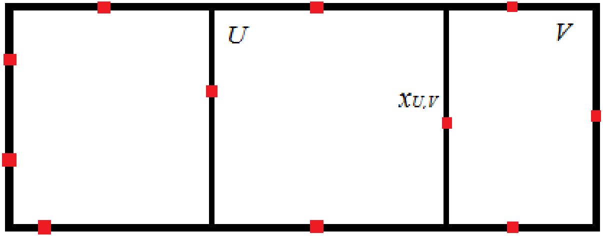
Constructing a convex metric

- Let $\mathcal{U}_1$ be a finite brick partition of X of mesh less than 1.
- Let N_1 be a finite connected graph in X such that:

  1.1 If $U \in \mathcal{U}_1$ then $N_1 \cap U$ is a non-empty tree with $N_1 \cap \mathrm{bd}(U)$ being the set of endpoints of $N_1 \cap U$.

  1.2 If $U, V \in \mathcal{U}_1$ are adjacent then $N_1 \cap U \cap V = \{x_{U,V}\} \subset \mathrm{int}(U \cup V)$.

  1.3 $\mathcal{U}_1|_{N_1}$ is a brick partition of N_1.



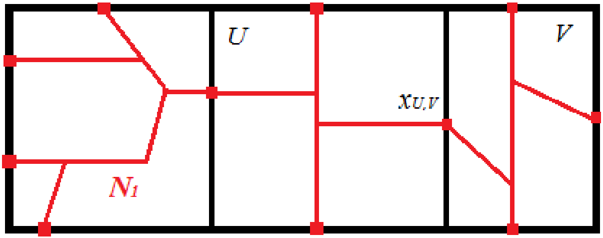
Constructing a convex metric

- Let $\mathcal{U}_1$ be a finite brick partition of X of mesh less than 1.
- Let N_1 be a finite connected graph in X such that:

  1.1 If $U \in \mathcal{U}_1$ then $N_1 \cap U$ is a non-empty tree with $N_1 \cap \mathrm{bd}(U)$ being the set of endpoints of $N_1 \cap U$.

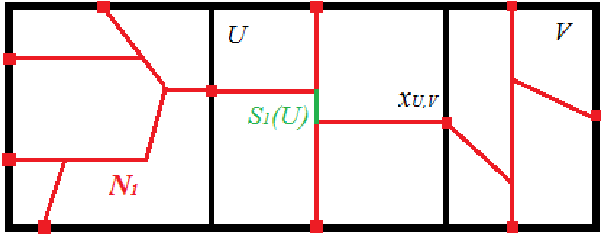
  1.2 If $U, V \in \mathcal{U}_1$ are adjacent then $N_1 \cap U \cap V = \{x_{U,V}\} \subset \mathrm{int}(U \cup V)$.

  1.3 $\mathcal{U}_1|_{N_1}$ is a brick partition of N_1.

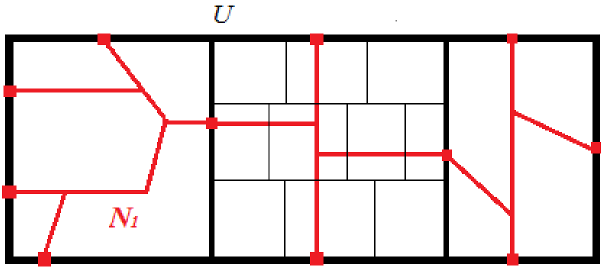


Constructing a convex metric

- For each $U \in \mathcal{U}_1$ let $S_1(U)$ be the smallest tree in $N_1 \cap U$ which contains all of the branch points of $N_1 \cap U$.
- Let $\mathcal{U}_2$ be a simultaneous core brick partition of X and N_1 of sufficiently small mesh which refines $\mathcal{U}_1$.

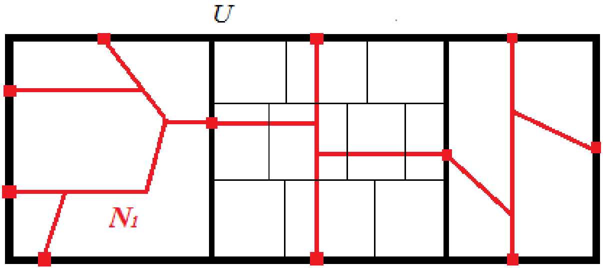


- For each $U \in \mathcal{U}_1$ let $S_1(U)$ be the smallest tree in $N_1 \cap U$ which contains all of the branch points of $N_1 \cap U$.
- Let $\mathcal{U}_2$ be a simultaneous core brick partition of X and N_1 of sufficiently small mesh which refines $\mathcal{U}_1$.



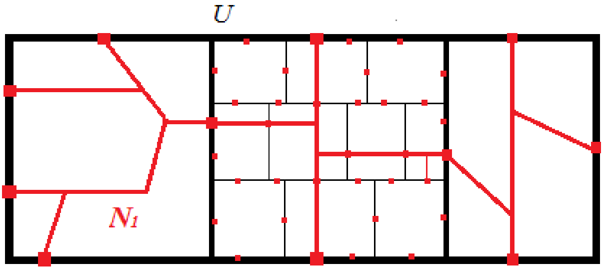
Constructing a convex metric

- Let N_2 be a finite graph in X such that:
  2.1 $\forall\ W \in \mathcal{U}_2$, $W \cap N_i$ is a tree with $\mathrm{bd}(W) \cap N_i$ as its endpoint, $i \in \{1, 2\}$ and $W \cap N_2 \neq \varnothing$.
  2.2 $\forall\ W, Z \in \mathcal{U}_2$ adjacent, $W \cap Z \cap N_2 = \{x_{W,Z}\} \subset \mathrm{int}(W \cup Z)$.
  2.3 $N_1 \subset N_2$.
  2.4 $\mathcal{U}_2|_{N_2}$ is a core brick partition of N_2.



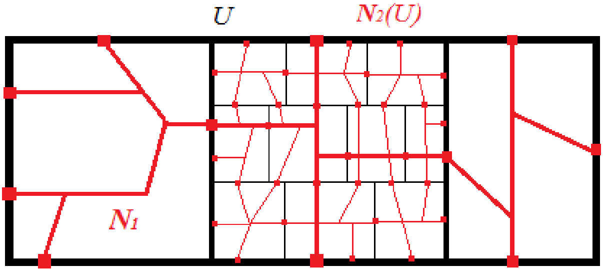
Constructing a convex metric

- Let N_2 be a finite graph in X such that:
  2.1 $\forall\ W \in \mathcal{U}_2$, $W \cap N_i$ is a tree with $\mathrm{bd}(W) \cap N_i$ as its endpoint, $i \in \{1, 2\}$ and $W \cap N_2 \neq \varnothing$.
  2.2 $\forall\ W, Z \in \mathcal{U}_2$ adjacent, $W \cap Z \cap N_2 = \{x_{W,Z}\} \subset \mathrm{int}(W \cup Z)$.
  2.3 $N_1 \subset N_2$.
  2.4 $\mathcal{U}_2|_{N_2}$ is a core brick partition of N_2.



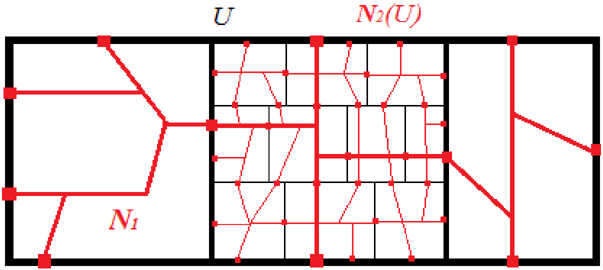
Constructing a convex metric

- Let N_2 be a finite graph in X such that:
  2.1 $\forall\ W \in \mathcal{U}_2$, $W \cap N_i$ is a tree with $\mathrm{bd}(W) \cap N_i$ as its endpoint, $i \in \{1, 2\}$ and $W \cap N_2 \neq \varnothing$.
  2.2 $\forall\ W, Z \in \mathcal{U}_2$ adjacent, $W \cap Z \cap N_2 = \{x_{W,Z}\} \subset \mathrm{int}(W \cup Z)$.
  2.3 $N_1 \subset N_2$.
  2.4 $\mathcal{U}_2|_{N_2}$ is a core brick partition of N_2.



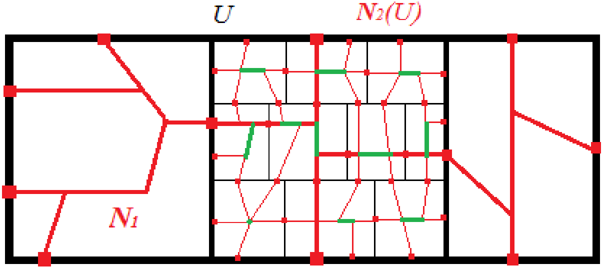
Constructing a convex metric

- For each $W \in \mathcal{U}_2$ let $S_2(W)$ be the smallest tree in $W \cap N_2$ which contains all of the branch points of $N_2 \cap W$.

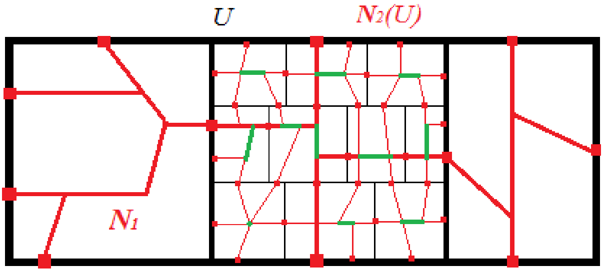


Constructing a convex metric

- For each $W \in \mathcal{U}_2$ let $S_2(W)$ be the smallest tree in $W \cap N_2$ which contains all of the branch points of $N_2 \cap W$.

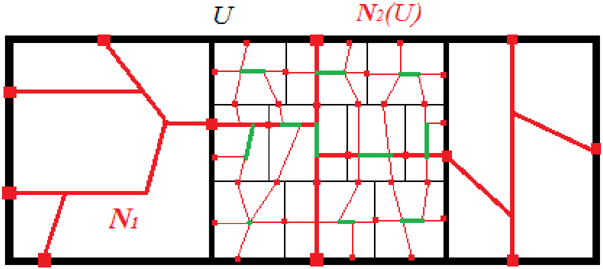


- Let $C_2(U) = \bigcup\{N_2(U) \cap \mathrm{bd}(W) \mid W \in \mathcal{U}_2(U)\} \cup \{x \in N_2(U) \mid x \text{ is a branch point of } N_2\}$.
- For each $U \in \mathcal{U}_1$ we define a partial metric ρ_1 on $C_2(U)$ which we eventually extend to a convex metric on X.



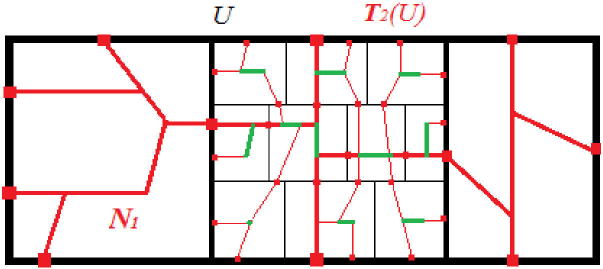
Constructing a convex metric

- For each $U \in \mathcal{U}_1$ let $T_2(U)$ be a maximal tree in $N_2 \cap U$ which contains $N_1 \cap U$, meets $\mathrm{int}(W)$ for each $W \in \mathcal{U}_2(U)$, and contains $\mathrm{bd}(U) \cap N_2$.
- Suppose that each component K of $(N_2 \cap U) \setminus T_2(U)$ is a maximal free arc in $N_2 \cap U$ and $T_2(U) \cup K$ contains a simple closed curve.
- Let $T_2 = \bigcup_{U \in \mathcal{U}_1} T_2(U)$.



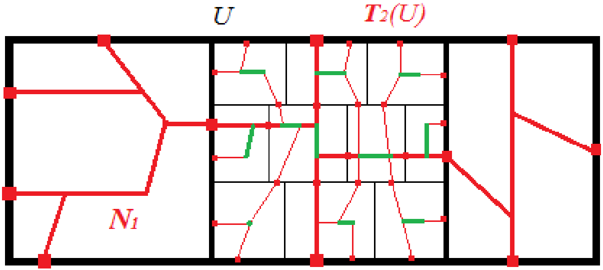
Constructing a convex metric

- For each $U \in \mathcal{U}_1$ let $T_2(U)$ be a maximal tree in $N_2 \cap U$ which contains $N_1 \cap U$, meets $\mathrm{int}(W)$ for each $W \in \mathcal{U}_2(U)$, and contains $\mathrm{bd}(U) \cap N_2$.
- Suppose that each component K of $(N_2 \cap U) \setminus T_2(U)$ is a maximal free arc in $N_2 \cap U$ and $T_2(U) \cup K$ contains a simple closed curve.
- Let $T_2 = \bigcup_{U \in \mathcal{U}_1} T_2(U)$.



Constructing a convex metric

- The approximating metric ρ_1 is constructed with conditions that will force to go along the graph T_2.
- The process is iterated.



Questions

- **Problem.** Suppose that X is a connected and locally arc-connected, separable, metric space which admits locally finite covers by small open and connected sets. Does X admit a convex metric generating its topology?
- **Problem.** Can the Theorem of Nikiel, Tuncali, Tymchatyn and S. be extended to the case of spaces admitting infinite partitions?

References

- R.H.Bing, *Partitioning a set*, Bull. Amer. Math. Soc. **55** (1949), 1101–1110.
- R.H.Bing, *Partitioning continuous curves*, Bull. Amer. Math. Soc. **58** (1952), 536–556.
- A.Chigogidze, K.Kawamura and E.D.Tymchatyn, *Nöbeling spaces and pseudo-interiors of Menger compacta*, Topology Appl. **68** (1996), 33-65.
- V.V.Filippov, *On locally connected spaces*, Dokl. Akad. Nauk SSSR **215** (1974), 666–668.
- E.E.Moise, *Grille decomposition and convexification theorems for compact metric locally connected continua*, Bull. Amer. Math. Soc. **55** (1949), 1111–1121.
- J. Nikiel, I. Stasyuk, M. Tuncali, E.D. Tymchatyn, *Convex metrics on non-compact spaces*, submitted to Fund. Math.
- I.Stasyuk, E.D. Tymchatyn, *Simultaneous brick partitionings of Peano continua* (preprint).
- G.H.M.Thomas, *Simultaneous partitionings of two sets*, Trans. Amer. Math. Soc. **75** (1953), 69–79.

THANK YOU !