ON CONTINUOUS LINEAR OPERATORS EXTENDING METRICS

Ihor Stasyuk, Edward Tymchatyn

Ivan Franko National University of Lviv, Lviv, Ukraine

University of Saskatchewan, Saskatoon, Canada

(X,d) – a complete metric space with $|X| \geq 2$.

$\exp_b(X) = \{A \subset X \mid A \text{ closed and bounded}\}$ with the Hausdorff metric H.

$\mathcal{PM}(A)$ (respectively $\mathcal{M}(A)$) – the family of all bounded, continuous pseudometrics (respectively metrics) on A, $A \in \exp_b(X)$.

$$\mathcal{PM} = \bigcup\{\mathcal{PM}(A) : A \in \exp_b(X)\},$$

$$\mathcal{M} = \bigcup\{\mathcal{M}(A) : A \in \exp_b(X),\ |A| \geq 2\}.$$

$\mathrm{dom}\rho = A$ if $\rho \in \mathcal{PM}(A)$,

$\|\rho\| = \sup\{\rho(x,y) \mid x,y \in \mathrm{dom}\rho\}$, $\rho \in \mathcal{PM}$.

Each pseudometric $\rho \in \mathcal{PM}$ is identified with its graph

$$\rho \to \Gamma_\rho = \{(x,y,\rho(x,y)) : x,y \in \mathrm{dom}\rho\} \in \exp_b(X \times X \times \mathbb{R})$$

The set $\exp(X \times X \times \mathbb{R})$ is equipped with the Hausdorff metric $\tilde{H}$ generated by the box metric on $X \times X \times \mathbb{R}$.

$$\mathcal{PM} \subset \exp_b(X \times X \times \mathbb{R}, \tilde{H}), \quad \mathcal{M} \subset \mathcal{PM}$$

Theorem (E.D. Tymchatyn and M. Zarichnyi, 2002)**.** *Let (X, d) be a compact metric space. There exists an operator $u\colon \mathcal{PM} \to \mathcal{PM}(X)$ that satisfies the following conditions for every $\rho, \rho' \in \mathcal{PM}$ and $c, c' > 0$:*

1) *$u(\rho)$ is an extension of ρ over X;*

2) *u is linear i.e. $u(c\rho + c'\rho') = cu(\rho) + c'u(\rho')$ whenever $\mathrm{dom}\rho = \mathrm{dom}\rho'$;*

3) *u is regular i.e. $\|u(\rho)\| = \|\rho\|$;*

4) *u is continuous.*

Theorem (E.D. Tymchatyn and M. Zarichnyi, 2002)**.** *There exists a regular extension operator $\tilde{u}\colon \mathcal{PM} \to \mathcal{PM}(X)$ such that $\tilde{u}(\mathcal{M}) \subset \mathcal{M}(X)$.*

Theorem 1. *Let (X, d) be a bounded, complete metric space. There exists an operator $w\colon \mathcal{PM} \to \mathcal{PM}(X)$ with the following properties for every $\rho, \rho' \in \mathcal{PM}$ and $c, c' > 0$:*

1) *$w(\rho)$ is an extension of ρ over X;*

2) *w is linear i.e. $w(c\rho + c'\rho') = cw(\rho) + c'w(\rho')$ whenever $\mathrm{dom}\rho = \mathrm{dom}\rho'$;*

3) *w is regular i.e. $\|w(\rho)\| = \|\rho\|$;*

4) *w is continuous with respect to the topology of uniform convergence on compact sets on $\mathcal{PM}(X)$ i.e. if $\{\rho_n\}$ converges to ρ in $\mathcal{PM}$ then $w(\rho_n)$ converges uniformly to $w(\rho)$ on compact subsets of $(X \times X)$.*

Comments on proof. $(T, \mathcal{A}, \mu)$ – a measurable space, $(E, \|\cdot\|)$ – a Banach space, $L_1(T, E)$ – the space of Bochner integrable functions with norm $\|\alpha\| = \int_0^1 \|\alpha(t)\| d\mu$.

A set D of measurable functions from $(T, \mathcal{A}, \mu)$ into a space Y is called *decomposable* if for every $\alpha, \beta \in D$ and $C \in \mathcal{A}$ the map $\alpha\chi_C + \beta\chi_{T\setminus C}$ belongs to D.

Theorem (S. Ageev and D. Repovs, 2000)**.** *Let* $(T, \mathcal{A}, \mu)$ *be a separable, measurable space,* E *a Banach space,* Y *a paracompact space. Then every lower semicontinuous, multi-valued map* $G\colon Y \to L_1(T, E)$ *with closed, decomposible values admits a continuous selection.*

Define a multi-valued map $F\colon X \times \exp_b(X) \to L_1(I, E)$ by

$$F(x, A) = \begin{cases} \{x\} = L_1(I, \{x\}) \text{ if } x \in A; \\ \\ L_1(I, A) \text{ if } x \notin A. \end{cases}$$

F admits a continuous single-valued selection $f\colon X \times \exp_b(X) \to L_1(I, E)$.

Let $w\colon \mathcal{PM} \to \mathcal{PM}(X)$ be given by the formula

$$w(\rho)(x, y) = \int_0^1 \rho(f(x, \mathrm{dom}\rho)(t), f(y, \mathrm{dom}\rho)(t))dt$$

for every $\rho \in \mathcal{PM}$ and $x, y \in X$.

□

Theorem 2. *Let (X, d) be a complete metric space. For arbitrary $\eta > 0$ there exists a linear, extension operator $v\colon \mathcal{M} \to \mathcal{M}(X)$ which is continuous with respect to the topology of uniform convergence on compact sets on $\mathcal{M}(X)$ and $\|v(\rho)\| \leq (1+\eta)\|\rho\|$ for every $\rho \in \mathcal{M}$.*

Comments on proof. $K = \{(x, A) \in X \times \exp_b(X) \mid x \in A\}$,

$q\colon X \times \exp_b(X) \to (X \times \exp_b(X))/K$ – quotient map.

Define a metric r on $(X \times \exp_b(X))/K$ by

$$r([(x, A)], [(y, B)]) = \min\{\min\{d(x, y) + H(A, B), d(x, A) + d(y, B)\}, 1\}.$$

There exists a continuous map $\gamma\colon \mathcal{M} \to (0, \infty)$ such that γ is linear i.e. $\gamma(t_1\rho_1 + t_2\rho_2) = t_1\gamma(\rho_1) + t_2\gamma(\rho_2)$ for every $t_1, t_2 > 0$ and $\rho_1, \rho_2 \in \mathcal{M}$ with $\mathrm{dom}\rho_1 = \mathrm{dom}\rho_2$.

For $\eta > 0$ let $v\colon \mathcal{M} \to \mathcal{M}(X)$ be defined as follows:

$$v(\rho)(x, y) = w(\rho)(x, y) + \eta \cdot \gamma(\rho) \cdot r(q(x, \mathrm{dom}\rho), q(y, \mathrm{dom}\rho))$$

for every $\rho \in \mathcal{M}$ and $x, y \in X$ where w is the map constructed in Theorem 1.

$\square$