Spaces of finite Hausdorff measure

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Definition 1. *Let* (X, ρ) *be a metric space and* $\alpha \geq 0$*. For any subset* A *of* X *and* $\delta > 0$ *let*

$$\mathcal{H}_\delta^\alpha(A, \rho) = \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}_\rho^\alpha(U_i) \mid A \subset \bigcup_{i=1}^{\infty} U_i \subseteq X,\ \operatorname{diam}_\rho(U_i) < \delta,\ i \in \mathbb{N} \right\}.$$

Then the α*-dimensional Hausdorff measure* $\mathcal{H}^\alpha$ *on* X *is defined by*

$$\mathcal{H}^\alpha(A, \rho) = \sup_{\delta > 0} \mathcal{H}_\delta^\alpha(A, \rho) = \lim_{\delta \to 0} \mathcal{H}_\delta^\alpha(A, \rho).$$

$\mathcal{H}^1$ *is called the linear Hausdorff measure on* X*.*

Proposition 1. $H^\alpha(A,\rho) < \infty \Longrightarrow H^\beta(A,\rho) = 0$ *for every* $\beta > \alpha$.

$H^\alpha(A,\rho) > 0 \Longrightarrow H^\beta(A,\rho) = \infty$ *for every* $0 < \beta < \alpha$.

Definition 2. *The Hausdorff dimension of the space* X *is defined by*

$$\dim_H(X) = \inf\{\alpha \geq 0 \mid \mathcal{H}^\alpha(X,\rho) = 0\}.$$

It is known that $\dim X \leq \dim_H(X)$.

Problem (Eilenberg, Harrold, 1943)**.** To characterize a separable metric space X with $\dim X = n$ for which there is a metrization ρ such that $\mathcal{H}^n(X,\rho) < \infty$.

If $\mathcal{H}^1(X,\rho) < \infty$ then (X,ρ) is said to be of finite linear measure. It is known that a continuum X admits a metric ρ such that (X,ρ) is of finite linear measure if and only if X is totally regular (meaning that for each countable set $P \subset X$ there is a basis $\mathcal{B}$ of open sets for X so that for each $B \in \mathcal{B}, P \cap bd(B) \neq \varnothing$ and B has finite boundary).

Theorem 1 (Nikiel, 1989)**.** *Suppose that a continuum X is homeomorphic to the inverse limit of connected graphs under monotone bonding surjections. Then X is totally regular.*

Theorem 2 (Buskirk, Nikiel, Tymchatyn, 1992)**.** *If X is a totally regular continuum, then there exists an inverse sequence $(X_i, \pi_{i,i+1})$ such that*

1) each X_i is a connected graph.

2) each $\pi_{i,i+1}\colon X_{i+1} \to X_i$ is a monotone surjection.

3) X is homeomorphic to $\varprojlim(X_i, \pi_{i,i+1})$.

4) $\pi_{i,i+1}^{-1}(x)$ is nondegenerate for exactly one point $x = x_0^i \in X_i$, $i \in \mathbb{N}$.

Theorem 3 (Buskirk, Nikiel, Tymchatyn, 1992)**.** *If X is totally regular then there exists a convex metric ρ on X such that $\mathcal{H}^1(X, \rho) < \infty$.*

Sketch of proof. For each polygonal arc ξ in $\mathbb{R}^3$ let $l(\xi)$ denote its length. We assume that $X = \varprojlim(X_i, \pi_{i,i+1})$ where each X_i is a connected graph and every bonding map is a monotone surjection. Moreover, we may suppose that $\pi_{i,i+1}^{-1}(x)$ is nondegenerate for exactly one point $x = x_0^i \in X_i$. We may assume also that $X_1 = \{x_0^1\}$ and that for every $i \in \mathbb{N}$ the following holds:

1) $X_i \subset \mathbb{R}^3$ and each subarc of X_i is a polygonal arc in $\mathbb{R}^3$,

2) if $\xi \subset X_{i+1}$ is an arc such that $x_0^i \notin \pi_{i,i+1}(\xi)$ then $l(\xi) = l(\pi_{i,i+1}(\xi))$,

3) for each triangulation K of $\pi_{i,i+1}^{-1}(x_0^i)$ we have $\sum_{j=1}^{m_i} l(\xi_j) < \frac{1}{2^i}$, where $\{\xi_1, \ldots, \xi_{m_i}\}$ is an enumeration of all edges of $(\pi_{i,i+1}^{-1}(x_0^i), K)$.

For $u, v \in X_i$ let

$$d_i(u, v) = \min\{l(\xi) \mid \xi \subset X_i \text{ is an arc with endpoints } u \text{ and } v\}.$$

Then for $s = (s_1, s_2, \dots)$ and $t = (t_1, t_2, \dots)$ from X the metric d defined by

$$d(s, t) = \sup\{d_i(s_i, t_i) \mid i \in \mathbb{N}\}$$

is convex and $\mathcal{H}^1(X, d) \leq 1$. $\square$

Definition 3. *A polyhedron is a compact metrizable space which is homeomorphic to the carrier $|K|$ of a finite simplicial complex K in a Euclidean space.*

For an integer $n > 1$ one can use the Buskirk-Nikiel-Tymchatyn construction to prove that $\mathcal{H}^n(X, \rho) < \infty$ for some metric ρ on X provided X is homeomorphic to the limit of an inverse sequence of n-dimensional polyhedra with monotone, surjective bonding maps of finite or countable rank.

Theorem 4. *Let* $X = \varprojlim(X_i, \pi_{i,i+1})$ *where each* X_i *is a connected polyhedron of dimension* n *and each* $\pi_{i,i+1}$ *is a monotone surjection of finite rank. Then there is a convex metric* ρ *on* X *such that* $\mathcal{H}^n(X, \rho) < \infty$*.*

Proposition 2. *Let* $\psi\colon (X,\rho) \to (Y,r)$ *be a surjective Lipschitz map between the metric spaces* X *and* Y *i.e. there is* $c > 0$ *such that* $r(\psi(x),\psi(y)) \leq c\rho(x,y)$ *for all* $x, y \in X$. *Then* $\mathcal{H}^\alpha(Y,r) \leq c^\alpha \mathcal{H}^\alpha(X,\rho)$.

So the Hausdorff dimension is a bi-Lipschitz invariant.

Definition 4. *Let* $\varepsilon > 0$, (X,ρ) *a compact metric space,* Y *a topological space. A continuous, surjective map* $f\colon X \to Y$ *is called an* ε*-map if* $\mathrm{diam}_\rho f^{-1}(y) \leq \varepsilon$ *for every* $y \in Y$.

Definition 5. *Let* $\mathcal{U}$ *be a finite open cover of a compact metric space* X. *Let* $N(\mathcal{U})$ *be the geometric realization of the nerve of the cover* $\mathcal{U}$ *and let* $|N(\mathcal{U})|$ *be its carrier. A map* $g\colon X \to |N(\mathcal{U})|$ *is said to be canonical if* $f^{-1}(\mathrm{St}_{N(\mathcal{U})}(U)) \subset U$ *for every vertex* $U \in \mathcal{U}$ *of* $N(\mathcal{U})$ *where* $\mathrm{St}_{N(\mathcal{U})}(U)$ *denotes the open star in* $N(\mathcal{U})$ *about the vertex* U. *If the mesh of the cover* $\mathcal{U}$ *is less or equal to* ε *then the canonical map* g *from* X *onto* $g(X) \subset N(\mathcal{U})$ *is an* ε*-mapping.*

Example 1. Suppose (X, ρ) is a compact metric space and let $\mathcal{U} = \{U_1, \dots, U_k\}$ be a finite open cover of X. Then the map $f\colon X \to |N(\mathcal{U})|$ defined for every $x \in X$ by the formula

$$f(x) = \sum_{i=1}^{k} t_i(x) U_i \text{ where } t_i(x) = \frac{\rho(x, X \setminus U_i)}{\sum_{j=1}^{k} \rho(x, X \setminus U_j)}, \; i \in \{1, \dots, k\}$$

is a canonical mapping from X into the carrier of the nerve of the cover $\mathcal{U}$.

Definition 6. *Let (X, ρ) be a metric space and let $\mathcal{U}$ be an open cover of X. The Lebesgue number of the cover $\mathcal{U}$ is*

$$\mathcal{L}(\mathcal{U}) = \inf_{x \in X} \{\max\{\rho(x, X \setminus U) \mid U \in \mathcal{U} \,|\}\}.$$

Proposition 3. *(Bell, Dranishnikov, 2002) For every n and every $\varepsilon > 0$ there exists a number $\nu = \nu(\varepsilon, k)$ such that for every cover $\mathcal{U}$ of X of order $\leq n+1$ with Lebesgue number $\mathcal{L}(\mathcal{U}) > \nu$ the canonical map $f\colon X \to |N(\mathcal{U})|$ defined above is ε-Lipschitz (it is enough to take $\nu \geq (2n+3)^2/\varepsilon$).*

Theorem 5 (Freudenthal, 1937)**.** *Every n-dimensional compact metric space (X,ρ) is the limit of an inverse sequence $(|K_i|,\pi_{i,i+1})$ of n-dimensional polyhedra $|K_i|$ such that each $\pi_{i,i+1}\colon |K_{i+1}|\to|K_i|$ is continuous and surjective. The polyhedra $|K_i|$ are given in fixed triangulations K_i and every map $\pi_{i,i+1}$ is a simplicial map of the triangulation K_{i+1} onto some multiple barycentric subdivision K_i^* of the triangulation K_i. Every projection $\pi_i\colon X\to|K_i|$ essentialy covers each principal simplex of K_i.*

Sketch of the proof. By induction. Take an irreducible 1/2-cover ω_1 of X which will be of order $n+1$ and let f_1 be a canonical map of X to the carrier of $|K_1|$ of the nerve of the cover ω_1. Then f_1 will essentially cover the principal simplices of K_1 and thus will be surjective. Suppose that for every $i<m$ we have constructed

1) n-dimensional polyhedra $|K_i|$ taken in triangulations K_i which are the nerves of irreducible covers ω_i of X.

2) $1/2^i$-maps $f_i\colon X \to |K_i|$ essentially covering the principal simplices of the triangulations K_i and

3) the maps $\pi_{i-1,i}\colon |K_i| \to |K_{i-1}|$, $1 < i < m$ each of which is simplicial with respect to the triangulation K_i and some multiple barycentric subdivision K^*_{i-1} of the triangulation K_{i-1} such that

4) for every $x \in X$, $\pi_{i-1,i} \circ f_i(x)$ and $f_{i-1}(x)$ belong to one simplex of K^*_{i-1}.

5) the maps $\pi_{j,i}$, $j < i$ satisfy the inequalities $d(\pi_{j,i-1} \circ f_{i-1}, \pi_{j,i} \circ f_i) < 1/2^i$.

Let ω_m be an irreducible, finite, open $1/2^m$-covering of the space X inscribed into the cover $\mathcal{V}_{m-1}$ consisting of the preimages of the stars of the vertices of the triangulation K^*_{m-1} under the map f_{m-1}. Let K_m be the nerve of ω_m and let $f_m\colon X \to |K_m|$ be any canonical map.

By a theorem of Mardešić there is a map $\pi_{m-1,m}\colon |K_m| \to |K_{m-1}|$ which is simplicial with respect to the triangulations K^*_{m-1} and K_m such that $\pi_{m-1,m}\circ f_m(x)$ and $f_{m-1}(x)$ belong to the same simplex of K^*_{m-1} for every $x \in X$ and thus

$$d(f_{m-1}, \pi_{m-1,m} \circ f_m) < \frac{1}{2^m},\ d(\pi_{j,m-1} \circ f_{m-1}, \pi_{j,m} \circ f_m) < \frac{1}{2^m},\ j < m-1.$$

Since the covers ω_i are irreducible, the map f_m essentially covers the principal simplices of the triangulation K_m and $\dim |K_m| \leq n$. Inductively we obtain an inverse sequence of polyhedra with required properties. Then the map

$$h = (h_1, h_2, \dots)\colon (X, \rho) \to \varprojlim |K_i|$$

defined by

$$h_i = \lim_{j\to\infty} \pi_{i,j} \circ f_j$$

for each $i \in \mathbb{N}$ is a homeomorphism. $\square$

Definition 7. *The Assouad-Nagata dimension of a metric space* (Y, r) *is the minimal integer* $\dim_{AN} Y = n$ *such that there exists* $\eta \in (0, 1)$ *with the property that for every* $\alpha > 0$ *one can find an open covering* $\mathcal{U}$ *of* X *of multiplicity* $m(\mathcal{U}) \leq n + 1$*,* $mesh(\mathcal{U}) \leq \alpha$ *and the Lebesgue number* $L(\mathcal{U}) \geq \eta\alpha$*.*

It is known that always $\dim Y \leq \dim_H Y \leq \dim_{AN} Y$.

Theorem 6 (Luukkainen, 1998)**.** *Every separable metrizable space* Y *can be metrized by a totally bounded metric for which* $\dim Y = \dim_{AN} Y$*.*

Definition 8. *Suppose that* (Y, r) *and* (Y', r') *are metric spaces. A map* $h\colon (Y, r) \to (Y', r')$ *is called quasi symmetric if there exists a homeomorphism* $\gamma\colon [0, \infty) \to [0, \infty)$ *such that for every triple* $x, a, b \in Y$ *the condition* $r(x, a) \leq tr(x, b)$ *implies* $r'(h(x), h(a)) \leq \gamma(t) r'(h(x), h(b))$*.*

The Assouad-Nagata dimension is a quasi symmetry invariant.

Conjecture. Let $n \in \mathbb{N}$, $n > 1$ and let (X, ρ) be a compact metric space with $\dim(X) = \dim_H(X) = \dim_{AN}(X) = n$ (so $\mathcal{H}^n(X, \rho) < \infty$). Then there exists an inverse sequence $\{Q_i, \pi_{ij}\}$ of n-dimensional polyhedra in $\mathbb{R}^{2n+1}$ and a metric d on $\varprojlim Q_i$ defined by $d(t, s) = \max\{c_i d_i(t_i, s_i) \mid i \in \mathbb{N}\}$ for $t, s \in \varprojlim Q_i$ where c_i are constants and d_i is a Euclidean metric for each i such that X is homeomorphic to $\varprojlim Q_i$ and $\mathcal{H}^n(\varprojlim Q_i, d) < \infty$.