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ON A HOMOGENEOUS OPERATOR EXTENDING PARTIAL ULTRAMETRICS

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Recently, E.D. Tymchatyn and M.M. Zarichnyi constructed a continuous extension operator for partial ultrametrics on a zero-dimensional compact metrizable space. This extension operator preserves the maximum of two ultrametrics but fails to preserve the homogeneity. The aim of this note is to provide a continuous homogeneous operator that extends partial ultrametrics and preserves norms.

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Э.Д. Тымчатын и М.М. Заричный построили непрерывный оператор продолжения ультраметрик, заданных на замкнутых подмножествах нульмерного метризуемого компактного пространства. Этот оператор продолжения сохраняет максимум двух ультраметрик, но не является однородным. Цель этой статьи — построение непрерывного однородного оператора, продолжающего ультраметрики и сохраняющего норму.

1. Introduction. The problem of extension of metrics has a long history. Here we briefly recall some results in this direction. The first result on the topic was proved by Hausdorff [1]. C. Bessaga [2] systematically considered the problem of existence of operators that extend (pseudo)metrics and preserve some algebraic and topological properties of the space of (pseudo)metrics. The problem was solved affirmatively by T. Banakh [3] (see also the survey [4] for related results).

Recently, a general problem of simultaneous extension of partial (pseudo)metrics was considered in [5]. Note that S. Mazurkiewicz [6] was the first who considered simultaneous extensions of metrics defined on the nonempty closed subsets of a compact metrizable space. In [7] E.D. Tymchatyn and M. Zarichnyi considered the problem of simultaneous extension of partial ultrametrics defined on the nonempty closed subsets of a zero-dimensional compact metrizable space. The main result of [7] states that there exists a continuous extension operator for partial ultrametrics that preserves norms (and, in particular, is continuous), suprema, and the Assouad dimension. It turned out, however, that the extension operator constructed in [7] is not homogeneous. M. Zarichnyi formulated a problem of existence of a homogeneous operator that extends partial ultrametrics defined on closed subsets of a compact zero-dimensional metrizable space.

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In this note we prove that there exists a homogeneous extension operator for partial ultrametrics that preserves norms and suprema.

2. Preliminaries. A metric ρ on a set Y is called an *ultrametric* if $\rho(x,y) \le \max\{\rho(x,z), \rho(z,y)\}$ for all $x,y,z \in Y$. For a metrizable space X there exists a compatible ultrametric if and only if $\dim X = 0$ (see, e.g., [8]). Let X be a zero-dimensional compact space. Consider the space $\exp X$ of all nonempty compact subsets of X equipped with the Vietoris topology. This topology has as a base all sets of the form

$$\langle V_1,\dots,V_n\rangle = \left\{A \in \exp X : A \subset \bigcup_{k=1}^{n} V_k,\ A\cap V_k \ne \varnothing \text{ for all } k\right\},$$

where $V_1,\dots,V_n$ is a finite family of open subsets of X.

For a compatible metric d on X the Vietoris topology on $\exp X$ is generated by the Hausdorff metric d_H,

$$d_H(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\}.$$

For a nonempty compact subset A of X let $\mathcal{UM}(A)$ be the set of continuous ultrametrics defined on A and let

$$\mathcal{UM} = \bigcup\{\mathcal{UM}(A) : A \in \exp X,\ |A| \ge 2\}$$

be the set of all partial ultrametrics (we impose the condition on the cardinality of the domain in order to avoid trivialities). We assume that every ultrametric $\rho \in \mathcal{UM}$ is identified with its graph $\Gamma_\rho \in \exp(X\times X\times \mathbb{R})$. We write $\operatorname{dom}\rho = A$ if $\rho \in \mathcal{UM}(A)$. For every $\rho \in \mathcal{UM}$ let $\|\rho\| = \max\{\rho(x,y) : x,y \in \operatorname{dom}\rho\}$. For every $A \in \exp X$ the set $\mathcal{UM}(A)$ is closed under the operation of pointwise maximum, max: $\mathcal{UM}(A)\times\mathcal{UM}(A) \to \mathcal{UM}(A)$. Let $\max\{\rho_1,\rho_2\} = \rho_1 \vee \rho_2 \in \mathcal{UM}(A)$ for all $\rho_1,\rho_2 \in \mathcal{UM}(A)$. For all $c \in \mathbb{R}_+$, $A \in \exp X$ and $\rho \in \mathcal{UM}(A)$ we have $c\rho \in \mathcal{UM}(A)$. Thus, the set $\mathcal{UM}(A)$ is a positive cone for every $A \in \exp X$.

3. Main result.

Theorem 1. *There exists an operator $u\colon \mathcal{UM} \to \mathcal{UM}(X)$ with the following properties for every $\rho,\sigma \in \mathcal{UM}$:*

1) $\|u(\rho)\| = \|\rho\|$;
2) *$u(\rho)$ is an extension of ρ;*
3) *$u(\rho\vee\sigma) = u(\rho)\vee u(\sigma)$ and $u(c\rho) = cu(\rho)$ if* $\operatorname{dom}\rho = \operatorname{dom}\sigma$ *and $c>0$;*
4) *u is continuous.*

Proof. For the sake of completeness of our exposition, we repeat the construction from the proof of the main result of [7].

Consider a multivalued map $G\colon X\times\exp X \to X$ defined in the following way:

$$G(z,B) = \begin{cases} B, & \text{if } z \notin B,\\ \{z\}, & \text{if } z \in B.\end{cases}$$

To prove that the map G is lower semicontinuous we show that the set

$$\tilde{U} = \{(z, B) \in X \times \exp X : G(z, B) \cap U \neq \varnothing\}$$

is open whenever U is open in X. Take arbitrary $(z_0, B_0) \in \tilde{U}$. There are two possibilities.

a) Let $z_0 \notin B_0$. Then $G(z_0, B_0) = B_0$ and $B_0 \cap U \neq \varnothing$. There exist neighborhoods V of z_0 and W of B_0 in X with $V \cap W = \varnothing$. Then for every $(z, B) \in V \times \langle W, W \cap U\rangle$ we have $z \notin B$ and therefore $G(z, B) = B$. Since $B \cap (W \cap U) \neq \varnothing$, we obtain $(z, B) \in \tilde{U}$.

b) Let $z_0 \in B_0$. We have $G(z_0, B_0) = \{z_0\}$ and $z_0 \in U$. Then $U \times \langle X, U\rangle$ is a neighborhood of (z_0, B_0) and for every $(z, B) \in U \times \langle X, U\rangle$ we have $G(z, B) \cap U \neq \varnothing$, i.e. $(z, B) \in \tilde{U}$.

Now let $A \in \exp X$, $|A| \geq 2$ and $(x, y) \in (X \times X) \setminus \Delta_X$, where Δ_X denotes the diagonal of X. Take $a_x, a_y \in A$ with $a_x \neq a_y$ and $a_x = x$ ($a_y = y$) if and only if $x \in A$ ($y \in A$). Now define a multivalued map $F_{(x,y,A)} \colon X \times \exp X \to X$ by the formula:

$$F_{(x,y,A)}(z, B) = \begin{cases} B, & \text{if } (x, A) \neq (z, B) \neq (y, A) \text{ and } z \notin B, \\ \{z\}, & \text{if } (x, A) \neq (z, B) \neq (y, A) \text{ and } z \in B, \\ \{a_x\}, & \text{if } (z, B) = (x, A), \\ \{a_y\}, & \text{if } (z, B) = (y, A). \end{cases}$$

We claim that the map $F_{(x,y,A)}$ is lower semicontinuous. Take arbitrary open subset U of X. Since G is lower semicontinuous, the set

$$\tilde{U} = \{(z, B) \in X \times \exp X : G(z, B) \cap U \neq \varnothing\}$$

is open in $X \times \exp X$. Then the set

$$\{(z, B) \in X \times \exp X : F_{(x,y,A)}(z, B) \cap U \neq \varnothing\} = \tilde{U} \setminus \{(z, A) : z \in \{x, y\}, a_z \notin U\}$$

is open in $X \times X$.

Since the space X is zero-dimensional, so is the space $\exp X$ and, consequently, $X \times \exp X$ (see, e.g., [9]). Therefore, by the zero-dimensional Michael Selection Theorem, there exists a continuous selection $f_{(x,y,A)}$ of $F_{(x,y,A)}$ i.e. $f_{(x,y,A)}$ is a single-valued map and for all $(z, B) \in X \times \exp X$ we have $f_{(x,y,A)}(z, B) \in F_{(x,y,A)}(z, B)$. Since $f_{(x,y,A)}(x, A) \neq f_{(x,y,A)}(y, A)$, there exist neighborhoods W_A of A in $\exp X$ and $V_{(x,y)}$ of (x, y) in $X \times X$ such that for every $A' \in W_A$ and $(x', y') \in V_{(x,y)}$ we have

$$f_{(x,y,A)}(x', A') \neq f_{(x,y,A)}(y', A').$$

Let

$$K = \{(x, y, A) : A \in \exp X,\ |A| \geq 2,\ (x, y) \in (X \times X) \setminus \Delta_X\}.$$

The cover $\mathcal{W} = \{V_{(x,y)} \times W_A : (x, y, A) \in K\}$ of K contains a countable subcover

$$\mathcal{W}' = \{V_{(x_i,y_i)} \times W_{A_i} : i \in \mathbb{N} \cup \{0\}\}$$

of K. For every $\rho \in \mathcal{UM}$ and $x, y \in X$ let

$$w_i(\rho)(x, y) = \rho\left(f_{(x_i,y_i,A_i)}(x, \operatorname{dom} \rho), f_{(x_i,y_i,A_i)}(y, \operatorname{dom} \rho)\right),\ i \in \mathbb{N} \cup \{0\}.$$

Now let

$$u(\rho)(x, y) = \max\left\{\frac{1}{2^i} w_i(\rho)(x, y) : i \in \mathbb{N} \cup \{0\}\right\}.$$

We now show that $u(\rho) \in \mathcal{UM}(X)$ for every $\rho \in \mathcal{UM}$. It is obvious that $u(\rho)(x,y) \geq 0$ and $u(\rho)(x,y) = u(\rho)(y,x)$ for all $x, y \in X$. Now let $(x,y) \in (X \times X) \setminus \Delta_X$. There exists a number $k \in \mathbb{N} \cup \{0\}$ such that $(x, y, \mathrm{dom}\, \rho) \in V_{(x_k,y_k)} \times W_{A_k}$. Since

$$f_{(x_k,y_k,A_k)}(x, \mathrm{dom}\, \rho) \neq f_{(x_k,y_k,A_k)}(y, \mathrm{dom}\, \rho),$$

we obtain $2^{-k} w_k(\rho)(x,y) := q > 0$ and therefore, $u(\rho)(x,y) > 0$. We now show that u is a well defined map. There exists $m \in \mathbb{N} \cup \{0\}$ such that $2^{-i} < q/\|\rho\|$ for all $i > m$. Therefore,

$$\frac{1}{2^i} w_i(\rho)(x,y) \leq \frac{1}{2^i}\|\rho\| < q$$

for all $i > m$. We obtain

$$u(\rho)(x,y) = \max\left\{\frac{1}{2^i} w_i(\rho)(x,y) : i \in \{0, \ldots, m\}\right\}.$$

We now show that $u(\rho)$ satisfies the ultrametric inequality. Let $x, y, z \in X$. There exists a number $j \in \mathbb{N} \cup \{0\}$ such that

$$u(\rho)(x,y) = \max\left\{\frac{1}{2^i} w_i(\rho)(x,y) : i \in \{0, \ldots, j\}\right\},$$
$$u(\rho)(x,z) = \max\left\{\frac{1}{2^i} w_i(\rho)(x,z) : \ i \in \{0, \ldots, j\}\right\}$$

and

$$u(\rho)(y,z) = \max\left\{\frac{1}{2^i} w_i(\rho)(y,z) : i \in \{0, \ldots, j\}\right\}.$$

Then

$$u(\rho)(x,y) \leq \max\left\{\frac{1}{2^i} \max\left\{w_i(\rho)(x,z), w_i(\rho)(y,z)\right\} : i \in \{0, \ldots, j\}\right\} =$$
$$= \max\left\{\max\left\{\frac{1}{2^i} w_i(\rho)(x,z) : i \in \{0, \ldots, j\}\right\},\ \max\left\{\frac{1}{2^i} w_i(\rho)(y,z) : i \in \{0, \ldots, j\}\right\}\right\} =$$
$$= \max\{u(\rho)(x,z), u(\rho)(y,z)\}.$$

It is obvious that $u(\rho)$ is continuous on $X \times X$. Therefore, $u(\rho) \in \mathcal{UM}(X)$ for every $\rho \in \mathcal{UM}$.

It is easy to see that $u(c\rho) = cu(\rho)$ for every $c \in \mathbb{R}_+$ and that $\|u(\rho)\| \leq \|\rho\|$ for all $\rho \in \mathcal{UM}$. If $x, y \in \mathrm{dom}\, \rho$ then $f_{(x_i,y_i,A_i)}(x, A) = x$ and $f_{(x_i,y_i,A_i)}(y, A) = y$ for all $i \in \mathbb{N} \cup \{0\}$. Therefore, $w_i(\rho)(x,y) = \rho(x,y)$ for all i and we obtain $u(\rho)(x,y) = \rho(x,y)$. Thus $u(\rho)$ is an extension of ρ and $\|u(\rho)\| \geq \|\rho\|$ which implies $\|u(\rho)\| = \|\rho\|$.

Now let $\rho_1, \rho_2 \in \mathcal{UM}$ with $\mathrm{dom}\, \rho_1 = \mathrm{dom}\, \rho_2 = B$, and $f_{(x_i,y_i,A_i)} = f_i$ for all $i \in \mathbb{N} \cup \{0\}$. For every $l \in \mathbb{N} \cup \{0\}$ let $I(l) = \{0, \ldots, l\}$. Fix $x, y \in X$. There exist $i_1, i_2 \in \mathbb{N} \cup \{0\}$ such that

$$(u(\rho_1)\vee u(\rho_2))(x,y)=$$
$$=\max\left\{\max\left\{\frac{1}{2^i}w_i(\rho_1)(x,y):i\in I(i_1)\right\},\max\left\{\frac{1}{2^i}w_i(\rho_2)(x,y):i\in I(i_2)\right\}\right\}=$$
$$=\max\left\{\max\left\{\frac{1}{2^i}w_i(\rho_1)(x,y),\frac{1}{2^i}w_i(\rho_2)(x,y)\right\}:i\in I(\max\{i_1,i_2\})\right\}=$$
$$=\max\left\{\max\left\{\frac{1}{2^i}\rho_1(f_i(x,B),f_i(y,B)),\frac{1}{2^i}\rho_2(f_i(x,B),f_i(y,B))\right\}:i\in I(\max\{i_1,i_2\})\right\}=$$
$$=\max\left\{\frac{1}{2^i}\max\{\rho_1(f_i(x,B),f_i(y,B)),\rho_2(f_i(x,B),f_i(y,B))\}:i\in I(\max\{i_1,i_2\})\right\}=$$
$$\max\left\{\frac{1}{2^i}w_i(\rho_1\vee\rho_2)(x,y):i\in\mathbb{N}\cup\{0\}\right\}=u(\rho_1\vee\rho_2)(x,y).$$

Now we are going to show that the map u is continuous. Let ρ_n be a sequence in $\mathcal{UM}$ that converges to $\rho\in\mathcal{UM}$. Then dom $\rho_n\to$ dom ρ. Let dom $\rho_n=B_n$ and dom $\rho=B$. There exists a continuous ultrametric $\tilde\rho$ on X that extends ρ over $X\times X$ (take, e.g., $\tilde\rho=u(\rho)$). Let $\tilde\rho_n=\tilde\rho|_{B_n\times B_n}$. Since $B_n\to B$ in $\exp X$, we have $\tilde\rho_n\to\rho$. Therefore, $d_H(\Gamma_{\rho_n},\Gamma_{\tilde\rho_n})\to 0$. From this we obtain

$$\max_{x,y\in B_n}|\rho_n(x,y)-\tilde\rho_n(x,y)|=\max_{x,y\in B_n}|\rho_n(x,y)-\tilde\rho(x,y)|\to 0.$$

Take arbitrary $\varepsilon>0$. There exists $n_0\in\mathbb{N}$ such that for every $n>n_0$ we have

$$\frac{1}{2^i}\big|\rho(f_i(x,B),f_i(y,B))-\tilde\rho(f_i(x,B_n),f_i(y,B_n))\big|<\frac{\varepsilon}{2},\tag{i}$$
$$\frac{1}{2^i}\max_{x,y\in B_n}|\rho_n(x,y)-\tilde\rho_n(x,y)|<\frac{\varepsilon}{2}\tag{ii}$$

for all $x,y\in X$ and $i\in\mathbb{N}\cup\{0\}$. Inequality (i) takes place due to the uniform continuity and boundedness of $\tilde\rho$ and the uniform continuity of the maps $\{f_i:i\in\mathbb{N}\cup\{0\}\}$ on $X\times\exp X$. We obtain

$$\frac{1}{2^i}|w_i(\rho)(x,y)-w_i(\rho_n)(x,y)|=\frac{1}{2^i}\big|\rho(f_i(x,B),f_i(y,B))-\rho_n(f_i(x,B_n),f_i(y,B_n))\big|\le$$
$$\le\frac{1}{2^i}\big|\rho(f_i(x,B),f_i(y,B))-\tilde\rho(f_i(x,B_n),f_i(y,B_n))\big|+$$
$$+\frac{1}{2^i}\big|\tilde\rho(f_i(x,B_n),f_i(y,B_n))-\rho_n(f_i(x,B_n),f_i(y,B_n))\big|<\varepsilon$$

for all $x,y\in X$ and $i\in\mathbb{N}\cup\{0\}$. Thus, we have $|u(\rho)(x,y)-u(\rho_n)(x,y)|<\varepsilon$ for all $n>n_0$ and $x,y\in X$ which implies $u(\rho_n)\to u(\rho)$. □

4. Remarks and open questions. It does not follow from our construction that the extension operator preserves the Assouad dimension (see, e.g., [10] for the definition).

Question 1. In the conditions of Theorem 1, is there an extension operator that satisfies (1)–(4) and preserves the Assouad dimension?

In [11], the notion of logarithmic ratio, a metric invariant of zero-dimensional spaces, is introduced.

Question 2. Is there an operator extending partial (pseudo)metrics for a zero-dimensional compact metric space that preserves the logarithmic ratio?

Question 3. Does there exist an extension operator for partial ultrametrics whose restriction onto $\mathcal{UM}(A)$ is continuous with respect to the pointwise convergence topology for each $A \in \exp(X)$ and which satisfies the conditions of Theorem 1?

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