

EXTENSION OF FUNCTIONS AND METRICS WITH VARIABLE DOMAINS

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ABSTRACT. Let (X, d) be a complete, bounded, metric space. For a nonempty, closed subset A of X denote by $C^*(A \times A)$ the set of all continuous, bounded, real-valued functions on $A \times A$. Denote by

$$C^\dagger = \bigcup \{C^*(A \times A) \mid A \text{ is a nonempty closed subset of } X\}$$

the set of all partial, continuous and bounded functions. We prove that there exists a linear, regular extension operator from C^\dagger endowed with the topology of convergence in the Hausdorff distance of graphs of partial functions to the space $C^*(X \times X)$ with the topology of uniform convergence on compact sets. The constructed extension operator preserves constant functions, pseudometrics, metrics and admissible metrics. For a fixed, nonempty, closed subset A of X the restricted extension operator from $C^*(A \times A)$ to $C^*(X \times X)$ is continuous with respect to the topologies of pointwise convergence, uniform convergence on compact sets and uniform convergence considered on both $C^*(A \times A)$ and $C^*(X \times X)$.

1. INTRODUCTION

The problem of extending continuous functions has a long history and is fundamental in topology and analysis. Improvements to the Tietze extension theorem and its counterpart for metrics, the Hausdorff extension theorem [10], have been made by many authors. The Dugundji result [8] on continuous, linear extensions of partial, continuous functions defined on a closed subspace of a metric space has counterparts for metrics as well. Initially the problem of existence of continuous, linear operators extending (pseudo)metrics was raised and solved for some special cases by Bessaga [7] and was completely solved by the first named author [2] (see also [3] and [19]). In [3] the authors obtained a rather general result on linear, continuous extensions of all functions and metrics defined on a stratifiable space with a fixed domain.

Further generalizations of known results on extensions of functions are related to the problem of simultaneous extension of partial functions with variable domains. Stepanova [17] obtained a non-linear, continuous extension operator for continuous, real-valued functions defined on variable, compact subsets of a metric space. She proved that the restriction onto metric spaces is essential. Künzi and Shapiro [12] improved Stepanova's result by showing existence of continuous, linear and regular extension operators under the same assumptions. A generalization of the

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Künzi-Shapiro result for the noncompact domain case was obtained in [11] (see also [4]). The third and the fourth named authors constructed an analogue of the Künzi-Shapiro theorem for (pseudo)metrics [18]. They described a linear, regular operator extending continuous (pseudo)metrics defined on closed subsets of a compact metrizable space. This operator is continuous with respect to the Hausdorff metric topology on the set of partial (pseudo)metrics where every (pseudo)metric is identified with its graph. Note that the Hausdorff metric convergence of graphs of continuous functions with common domain implies pointwise convergence as well as uniform convergence on compact sets but does not imply the uniform convergence of these functions. However, if the limit function is uniformly continuous then this last implication is true (see [5] and [14]). Of course all the metrics considered in [18] are uniformly continuous because the initial space is compact, so the convergence of graphs of metrics defined on the whole space is equivalent to the uniform convergence.

In [16] the second and the third named authors obtained a counterpart of the main result from [18] for noncompact, complete, metric spaces X . They proved existence of a linear, regular operator extending bounded, continuous pseudometrics defined on closed, bounded subsets of X . The set of partial pseudometrics was endowed with the Hausdorff metric topology and the set of pseudometrics defined on X was considered with the topology of uniform convergence on compact sets. It was shown that the operator is continuous. However, the authors were not able to modify this operator to preserve metrics without losing regularity.

In the current paper we construct an improvement of the operator from [16]. Using ideas from [3], [18] and [16] we describe an extension operator for partial, continuous, bounded functions and (pseudo)metrics on variable, closed subsets of a complete, metric space with a broad set of properties. In particular, we show that our operator is linear, regular and continuous in several important topologies. Another important feature of our operator is that it preserves admissible metrics, a fact that is not generally true for the operator in [16].

2. PRELIMINARIES

Let (X, d) be a complete, bounded, metric space with $\text{card}(X) \geq 2$. We assume that $d(x, y) \leq 1$ for all $x, y \in X$. Denote by $\text{exp}(X)$ the set of all nonempty, closed subsets of X endowed with the Hausdorff metric d_H generated by d . Recall that the Hausdorff distance between any sets $A, B \in \text{exp}(X)$ is given by

$$d_H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}.$$

Denote the set of singleton subsets of X by $\text{exp}_1(X)$.

For each $A \in \text{exp}(X)$ let $C^*(A \times A)$ be the set of all continuous and bounded real-valued functions on $A \times A$. Define

$$C^\dagger = \bigcup \{C^*(A \times A) \mid A \in \text{exp}(X)\}$$

to be the set of all partial, continuous, bounded, real-valued functions defined on closed subsets of X .

A (pseudo)metric ρ on $A \in \text{exp}(X)$ ($\text{card}(A) \geq 2$ for metrics) is called *continuous* if the identity map $\text{id}: (A, d|_{A \times A}) \rightarrow (A, \rho)$ is continuous.

A metric ρ on $A \in \exp(X)$ with $\text{card}(A) \geq 2$ is called *dominating* if the identity map $\text{id}: (A, \rho) \rightarrow (A, d|_{A \times A})$ is continuous. We note that dominating metrics need not be continuous.

A metric ρ on $A \in \exp(X)$ with $\text{card}(A) \geq 2$ that is both continuous and dominating is called *admissible*. It generates the initial topology of A as a subspace of X .

For every $A \in \exp(X)$ let $\mathcal{PM}(A)$ denote the set of all continuous and bounded pseudometrics on A .

For every $A \in \exp(X)$ with $\text{card}(A) \geq 2$ let $\mathcal{M}(A)$ (respectively, $\mathcal{M}_a(A)$) stand for the set of all continuous and bounded (respectively, admissible and bounded) metrics on A .

If p belongs to one of the sets $C^*(A \times A)$, $\mathcal{PM}(A)$, $\mathcal{M}(A)$ or $\mathcal{M}_a(A)$ we will write $\text{dom } p = A$. It is clear that for a fixed $A \in \exp(X)$ the classes of pseudometrics and metrics (if $\text{card}(A) \geq 2$) defined above are all positive cones in the sense that these sets are closed under the operations of pointwise addition and multiplication by a nonnegative (positive for metrics) number. Let

$$\mathcal{PM} = \bigcup \{ \mathcal{PM}(A) \mid A \in \exp(X) \}$$

be the set of all partial, continuous and bounded pseudometrics defined on closed subsets of X . Analogously, let

$$\mathcal{M} = \bigcup \{ \mathcal{M}(A) \mid A \in \exp(X), \text{card}(A) \geq 2 \}$$

be the set of all partial, continuous and bounded metrics on closed subsets of X . In a similar manner we define

$$\mathcal{M}_a = \bigcup \{ \mathcal{M}_a(A) \mid A \in \exp(X), \text{card}(A) \geq 2 \}$$

to be the set of all partial, admissible and bounded metrics on closed subsets of X .

We endow the set C^\dagger with the Hausdorff metric topology. We assume that every partial function $p \in C^\dagger$ is identified with its graph

$$\Gamma_p = \{ (x, y, p(x, y)) \mid x, y \in \text{dom } p \}$$

which is a closed and bounded subset of $X \times X \times \mathbb{R}$.

Let \tilde{d} be the l_1 metric on $X \times X \times \mathbb{R}$ defined by

$$\tilde{d}[(x, y, z), (x', y', z')] = d(x, x') + d(y, y') + |z - z'|$$

for $x, y, x', y' \in X$ and $z, z' \in \mathbb{R}$. We will also use the same symbol \tilde{d} to denote the l_1 metric on $X \times X$.

Let \tilde{d}_H denote the Hausdorff metric on $\exp(X \times X \times \mathbb{R})$ generated by \tilde{d} on $X \times X \times \mathbb{R}$. Then C^\dagger can be viewed as a subspace of the space $(\exp(X \times X \times \mathbb{R}), \tilde{d}_H)$ where the distance between any two partial functions from C^\dagger is the Hausdorff distance between their graphs. Then the sets \mathcal{PM} , \mathcal{M} and \mathcal{M}_a are subspaces of C^\dagger .

Let φ be the l_1 metric on $X \times \exp(X)$ defined by

$$\varphi[(x, A), (x', A')] = d(x, x') + d_H(A, A')$$

for all $x, x' \in X$ and $A, A' \in \exp(X)$.

For every $p \in C^\dagger$ let $\|p\|_\infty = \sup\{|p(x, y)| \mid x, y \in \text{dom } p\}$.

A map $u: C^\dagger \rightarrow C^*(X \times X)$ is called an *extension operator* if $u(p)|_{\text{dom } p \times \text{dom } p} = p$ for every $p \in C^\dagger$.

A map $u: C^\dagger \rightarrow C^*(X \times X)$ is called *linear* if $u(t_1p_1 + t_2p_2) = t_1u(p_1) + t_2u(p_2)$ for every $t_1, t_2 \in \mathbb{R}$ and $p_1, p_2 \in C^\dagger$ with $\text{dom } p_1 = \text{dom } p_2$.

A map $u: C^\dagger \rightarrow C^*(X \times X)$ is called *regular* if $\|u(p)\|_\infty = \|p\|_\infty$ for every $p \in C^\dagger$.

Let I denote the unit interval and let \mathbb{N} be the set of positive integers. We denote by \mathbb{R}^Y the set of all real-valued functions from a topological space Y .

We will use the symbols π_1 and π_2 to denote the projection functions from the space $X \times \text{exp}(X)$ onto the first and the second coordinates, respectively.

Our main result is the following:

Theorem 2.1. *There exists an operator $T: C^\dagger \rightarrow C^*(X \times X)$ with the following properties:*

- 1) T is an extension operator;
- 2) T is linear;
- 3) T is regular;
- 4) T preserves constant functions;
- 5) $T(\mathcal{PM}) \subset \mathcal{PM}(X)$;
- 6) $T(\mathcal{M}) \subset \mathcal{M}(X)$;
- 7) $T(\mathcal{M}_a) \subset \mathcal{M}_a(X)$;
- 8) T is continuous as a map from C^\dagger with the topology of convergence in the Hausdorff distance of graphs to $C^*(X \times X)$ with the topology of uniform convergence on compact sets;
- 9) for any fixed $A \in \text{exp}(X)$, $T|_{C^*(A \times A)}$ and $T|_{\mathcal{PM}(A)}$ are continuous with respect to the topologies of pointwise convergence, uniform convergence on compact sets and uniform convergence and, in particular,
- 10) for any fixed $A \in \text{exp}(X)$ with $\text{card}(A) \geq 2$, $T|_{\mathcal{M}(A)}$ and $T|_{\mathcal{M}_a(A)}$ are continuous with respect to the topologies of pointwise convergence, uniform convergence on compact sets and uniform convergence.

Remark. Note that $C^*(A)$ embeds in $C^*(A \times A)$ by the formula $p \mapsto \tilde{p}$ where $\tilde{p}(x, y) = \frac{1}{2}(p(x) + p(y))$ for each $p \in C^*(A)$ and $x, y \in A$. Hence, Theorem 2.1 generalizes both the Dugundji theorem for bounded, continuous functions and the Künzi-Shapiro result.

3. AUXILIARY RESULTS

For a measurable space (M, \mathcal{A}, μ) and a Banach space $(E, \|\cdot\|)$ consider the space $L_1(M, E)$ of Bochner integrable functions from M to E with the norm defined by $\|\alpha\|_1 = \int_0^1 \|\alpha(t)\| d\mu$ for every $\alpha \in L_1(M, E)$.

A set S of measurable functions acting from (M, \mathcal{A}, μ) into a topological space Y is called *decomposable* if for every $\alpha, \beta \in S$ and every $M' \in \mathcal{A}$ the map $\alpha\chi_{M'} + \beta\chi_{M \setminus M'}$ belongs to S (here χ denotes the characteristic function).

Let Y and Z be topological spaces. A multi-valued map $F: Y \rightarrow Z$ is called *lower semicontinuous* if the set $U^\sharp = \{y \in Y \mid F(y) \cap U \neq \emptyset\}$ is open in Y for every open subset U of Z .

We will use the following result due to Ageev and Repovš in our construction.

Theorem 3.1 ([1]). *Let (M, \mathcal{A}, μ) be a separable, measurable space, E a Banach space, Y a paracompact space and $L_1(M, E)$ the space of all Bochner integrable functions from M to E . Then every dispersible, multi-valued map $F: Y \rightarrow L_1(M, E)$ with closed values admits a continuous selection.*

We refer the reader to [1] for the definition of a dispersible multi-valued map and note that we will use the fact that every lower semicontinuous, multi-valued map with decomposable values is dispersible. So if additionally its values are closed, it satisfies the hypothesis of the above theorem (see [1]).

We will apply Theorem 3.1 in case where (M, \mathcal{A}, μ) is the unit interval I with the standard Lebesgue measure. Since the Lebesgue measure is regular the Bochner integral for functions with the domain I may be defined using step functions (see [13]).

Now we are going to prove an auxiliary property of Bochner integrable functions on I . For $I = [0, 1]$ with the standard Lebesgue measure μ and a Banach space E consider the space $L_1(I, E)$.

Proposition 3.2. *Let $\alpha \in L_1(I, E)$. Then for real numbers a and b with $0 \leq a \leq b \leq 1$ the transformation $\alpha \mapsto \alpha^{a,b} \in L_1(I, E)$ defined by the formula*

$$\alpha^{a,b}(t) = \begin{cases} \alpha\left(\frac{1}{b-a}(t-a)\right) & \text{if } a \leq t < b; \\ 0 & \text{elsewhere in } [0, 1]. \end{cases}$$

is continuous with respect to a and b . In other words, for any sequences $\{a_i\}, \{b_i\}$ such that $0 \leq a_i \leq b_i \leq 1$ for all $i \in \mathbb{N}$ and such that $a_i \rightarrow a$ and $b_i \rightarrow b$ as $i \rightarrow \infty$ we have

$$\|\alpha^{a,b} - \alpha^{a_i,b_i}\|_1 = \int_0^1 \|\alpha^{a,b}(t) - \alpha^{a_i,b_i}(t)\| dt \rightarrow 0 \text{ as } i \rightarrow \infty.$$

Proof. Fix any $\alpha \in L_1(I, E)$, any numbers a, b with $0 \leq a \leq b \leq 1$ and any sequences $\{a_i\}, \{b_i\}$ as specified in the hypothesis. Choose any $\varepsilon > 0$. One can find a function $\beta \in L_1(I, E)$ with finitely many values $z_1, \dots, z_k \in E$ with each preimage $\beta^{-1}(z_j)$ being a nondegenerate subinterval of I and such that $\|\alpha - \beta\|_1 < \varepsilon/4$. Let $\eta = \min\{\mu(\beta^{-1}(z_j)) \mid j \in \{1, \dots, k\}\} > 0$ and let $\eta' = \max\{\|z_j - z_m\| \mid j, m \in \{1, \dots, k\}\}$. Choose an index $i_0 \in \mathbb{N}$ large enough so that

$$|a - a_i| < \frac{\eta\varepsilon}{4k(\eta' + 1)} \text{ and } |b - b_i| < \frac{\eta\varepsilon}{4k(\eta' + 1)} \text{ for each } i \geq i_0.$$

Then for $i \geq i_0$ we obtain

$$\begin{aligned} \|\alpha^{a,b} - \alpha^{a_i,b_i}\|_1 &\leq \\ &\|\alpha^{a,b} - \beta^{a,b}\|_1 + \|\beta^{a,b} - \beta^{a_i,b_i}\|_1 + \|\beta^{a_i,b_i} - \alpha^{a_i,b_i}\|_1 = \\ &(b-a)\|\alpha - \beta\|_1 + \|\beta^{a,b} - \beta^{a_i,b_i}\|_1 + (b_i - a_i)\|\beta - \alpha\|_1 < \\ &\frac{\varepsilon}{4} + \int_0^1 \|\beta^{a,b}(t) - \beta^{a_i,b_i}(t)\| dt + \frac{\varepsilon}{4} < \sum_{j=1}^k 2\eta' \frac{\eta\varepsilon}{4k(\eta' + 1)} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

4. THE EXTENSION OPERATOR T

From now on we assume that (X, d) is embedded isometrically into a Banach space $(E, \|\cdot\|)$ as a closed subset. This subset is clearly bounded in E since (X, d) is. As before, I is the unit interval with the standard Lebesgue measure and for any $A \subset E$ let $L_1(I, A)$ be the set of all functions $\alpha \in L_1(I, E)$ such that $\alpha(I) \subset A$. If $x \in X \subset E$ we will write $x = x(t)$ or $\{x\}$ to denote the constant function $L_1(I, \{x\})$.

Let $K = \{A \times \{A\} \mid A \in \exp(X)\}$ which is a closed subspace of the space $(X \times \exp(X), \varphi)$. Consider an open, locally finite cover \mathcal{Q} of $(X \times \exp(X)) \setminus K$ such that $Q \in \mathcal{Q}$ implies that

$$\text{diam}_\varphi(Q) < \frac{1}{2}\varphi((a, A), Q) \text{ for each } (a, A) \in K.$$

For $Q \in \mathcal{Q}$ let

$$W_Q = \{(a, A) \in K \mid \text{there exists } (x, A) \in Q \text{ with } \varphi[(x, A), (a, A)] < 2\varphi[(x, A), A \times \{A\}]\}.$$

To show that W_Q is open in K let $(a, A) \in W_Q$ and let $(x, A) \in Q$ be such that $\varphi[(x, A), (a, A)] < 2\varphi[(x, A), A \times \{A\}]$. Now let (a_i, A_i) be a sequence in $X \times \exp(X)$ converging to (a, A) as $i \rightarrow \infty$. Note that $(x, A_i) \in Q$ for large i since Q is open. Since $\varphi[(x, A_i), A_i \times \{A_i\}] \rightarrow \varphi[(x, A), A \times \{A\}]$ and $\varphi[(x, A_i), (a_i, A_i)] \rightarrow \varphi[(x, A), (a, A)]$ as $i \rightarrow \infty$ we obtain $(a_i, A_i) \in W_Q$ for large i and so W_Q is open in K .

Define a multi-valued map $F: X \times \exp(X) \rightarrow L_1(I, E)$ by the formula

$$F(x, A) = \begin{cases} L_1(I, \{a \in A \mid (a, A) \in W_Q \text{ with } (x, A) \in Q \in \mathcal{Q}\}) & \text{if } x \notin A; \\ \{x\} = L_1(I, \{x\}) & \text{if } x \in A. \end{cases}$$

Proposition 4.1. *The map F is lower semicontinuous.*

Proof. We need to verify that for every open set S in $L_1(I, E)$ the set

$$S^\sharp = \{(x, A) \in X \times \exp(X) \mid F(x, A) \cap S \neq \emptyset\}$$

is open in $X \times \exp(X)$. So fix any open set $S \subset L_1(I, E)$ and any point $(x, A) \in S^\sharp$. Consider the following two cases.

Case 1. Let $x \notin A$. This means that $(x, A) \notin K$. Since \mathcal{Q} is locally finite, let Q_1, \dots, Q_k be all the elements from \mathcal{Q} that contain (x, A) . Let

$$\varepsilon_1 = \varphi[(x, A), (X \times \exp(X)) \setminus \cap_{i=1}^k Q_i] > 0.$$

There exists a function $\alpha \in F(x, A) \cap S$, a number $\varepsilon > 0$ and an ε -neighbourhood $O_\varepsilon(\alpha)$ of α in $L_1(I, E)$ such that $\alpha \in O_\varepsilon(\alpha) \subset S$. One can find a function $\beta \in F(x, A)$ that takes a finite number of values, say, b_1, \dots, b_m such that $\|\alpha - \beta\|_1 < \varepsilon/2$. Since $\cup_{i=1}^k W_{Q_i}$ is open in K there exists a number $\varepsilon_2 > 0$ such that for each $y \in X \subset E$ the inequality $\|b_j - y\| < \varepsilon_2$ implies that $y \in \pi_1\{\cup_{i=1}^k W_{Q_i}\}$. Let $\varepsilon' = \frac{1}{2} \min\{\varepsilon, \varepsilon_1, \varepsilon_2\}$. Find a neighbourhood V of (x, A) in $X \times \exp(X)$ such that $d(x, x') \equiv \|x - x'\| < \varepsilon'$ and $d_H(A, A') < \varepsilon'$ for all $(x', A') \in V$.

If $(x', A') \in V$ then $(x', A') \in \cap_{i=1}^k Q_i$ because $\varphi[(x, A), (x', A')] < 2\varepsilon' \leq \varepsilon_1$. Perturb the values of β to get a function $\beta' \in L_1(I, A')$ with the values $b'_1, \dots, b'_m \in A'$ and such that $\|b_j - b'_j\| < \varepsilon' \leq \varepsilon_2/2 < \varepsilon_2$ for all $j \in \{1, \dots, m\}$. We obtain

$$b'_1, \dots, b'_m \in \pi_1\{\cup_{i=1}^k W_{Q_i}\} \subset \{a' \in A' \mid (a', A') \in W_Q \text{ with } (x', A') \in Q \in \mathcal{Q}\}$$

and so $\beta' \in F(x', A')$. Since $\beta^{-1}(b_j) = \beta'^{-1}(b'_j)$ for each $j \in \{1, \dots, m\}$ we obtain $\|\beta - \beta'\|_1 < \varepsilon' \leq \varepsilon/2$ and so

$$\|\alpha - \beta'\|_1 \leq \|\alpha - \beta\|_1 + \|\beta - \beta'\|_1 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\beta' \in S$.

Case 2. Let $x \in A$. This means that $(x, A) \in K$. By the definition of F , $F(x, A) = \{x\}$ and so $\{x\} \in S$. Since S is open there exists a number $\varepsilon > 0$ and a neighbourhood $O_\varepsilon(\{x\})$ of the constant function $\{x\}$ in $L_1(I, E)$ such that $O_\varepsilon(\{x\}) \subset S$. Find a neighbourhood V of (x, A) in $X \times \exp(X)$ with $d(x, x') \equiv \|x - x'\| < \varepsilon$ and $d_H(A, A') < \varepsilon$ for all $(x', A') \in V$. Fix any $(x', A') \in V$. If $x' \in A'$ we obtain $F(x', A') = \{x'\} \in S$. If $x' \notin A'$, let $a' \in A'$ be such that $d(a', x) < \varepsilon$. Then $\{a'\} \in F(x', A') \cap S$ and so $(x', A') \in S^\#$. \square

Define a set-valued function $\overline{F} : X \times \exp(X) \rightarrow L_1(I, E)$ by setting $\overline{F}(x, A) = \overline{F(x, A)}$ for $(x, A) \in X \times \exp(X)$. Then, \overline{F} is also lower semicontinuous and has closed values in $L_1(I, E)$ (note that $(X \times \exp(X), \varphi)$ is a complete metric space, see [9, page 298, 4.5.23(c)]). For each $(x, A) \in X \times \exp(X)$, $\overline{F}(x, A)$ is the set of all functions from $L_1(I, E)$ whose values belong to the closure of the set

$$\{a \in A \mid (a, A) \in W_Q \text{ with } (x, A) \in Q \in \mathcal{Q}\}$$

in X which means that \overline{F} has decomposable values. Therefore \overline{F} is dispersible in the sense of [1] and so it admits a continuous, single-valued selection $f : X \times \exp(X) \rightarrow L_1(I, E)$ by Theorem 3.1.

We remark that the operator $w : C^\dagger \rightarrow \mathbb{R}^{X \times X}$ given by the formula

$$w(p)(x, y) = \int_0^1 p(f(x, \text{dom } p)(t), f(y, \text{dom } p)(t)) dt$$

for every $p \in C^\dagger$ and $x, y \in X$, preserves continuous, bounded functions and pseudometrics (compare with [16]). However, w does not necessarily map metrics to metrics and so we will not use w in this paper. Instead with the help of the selection f we will define a more complicated extension operator T that will have all the properties required to prove our main result.

In [3] the authors fixed a closed, non-singleton subset A of the original space so that all partial (pseudo)metrics and functions were defined on $A \times A$. They used two arbitrary distinct points from A in order to define the extension operator preserving metrics as a countable sum of operators separating points. In the current paper we deal with variable domains and in general it is impossible to choose continuously pairs of distinct points in each domain. To extend the idea from [3] we will use pairs of functions from $L_1(I, X)$ rather than pairs of points from X . Our arguments parallel those in [3].

Proposition 4.2. *There exist two continuous functions $g, h : \exp(X) \rightarrow L_1(I, X)$ such that $\int_0^1 \|g(B)(t) - h(B)(t)\| dt > 0$ for each $B \in \exp(X)$ with $\text{card}(B) \geq 2$ and $g(B) = h(B)$ if $\text{card}(B) = 1$.*

Proof. For each $A \in \exp(X)$ pick arbitrarily two points a_A and b_A from A such that $a_A \neq b_A$ if $\text{card}(A) \geq 2$. It is clear that $a_A = b_A$ if $\text{card}(A) = 1$. Define the multi-valued maps $G_A, H_A : \exp(X) \rightarrow L_1(I, E)$ as follows

$$G_A(B) = \begin{cases} L_1(I, B) & \text{if } A \neq B; \\ \{a_A\} = L_1(I, \{a_A\}) & \text{if } A = B \end{cases}$$

and

$$H_A(B) = \begin{cases} L_1(I, B) & \text{if } A \neq B; \\ \{b_A\} = L_1(I, \{b_A\}) & \text{if } A = B. \end{cases}$$

Note that $G_A = H_A$ if $\text{card}(A) = 1$.

Let us prove that for each $A \in \exp(X)$ the maps G_A and H_A are both lower semicontinuous. Fix any $A \in \exp(X)$. For an open subset S of $L_1(I, E)$ let $S^\# = \{B \in \exp(X) \mid G_A(B) \cap S \neq \emptyset\}$. Choose any $B \in S^\#$ and consider the following two cases.

Case 1. $B \neq A$. Then $G_A(B) = L_1(I, B)$. Since $G_A(B) \cap S \neq \emptyset$ there exists a function $\alpha \in G_A(B) \cap S$. One can find a number $\varepsilon > 0$ and an ε -neighbourhood $O_\varepsilon(\alpha)$ of α in $L_1(I, E)$ such that $O_\varepsilon(\alpha) \subset S$. There is a neighbourhood W of B in $\exp(X)$ such that $d_H(B, B') < \varepsilon/2$ and $B' \neq A$ for all $B' \in W$. There exists a function $\beta \in L_1(I, B)$ such that $\|\alpha - \beta\|_1 < \varepsilon/2$ and β takes a finite number of values. Let $B' \in W$. As in Proposition 4.1 we perturb the values of β to get a function $\beta' \in L_1(I, B')$ with the property that $\|\beta - \beta'\|_1 < \varepsilon/2$. We obtain $\|\alpha - \beta'\|_1 < \varepsilon$ and therefore, $\beta' \in S$. So $W \subset S^\#$.

Case 2. $B = A$. Then $G_A(B) = G_A(A) = \{a_A\}$. Since $\{a_A\} \in S$ there exists a number $\varepsilon > 0$ and a neighbourhood $O_\varepsilon(\{a_A\})$ of $\{a_A\}$ in $L_1(I, E)$ such that $O_\varepsilon(\{a_A\}) \subset S$. There is a neighbourhood W of A in $\exp(X)$ such that $d_H(A, A') < \varepsilon$ for all $A' \in W$. For $A' \in W$ denote by α' the constant map whose value is x' where $x' \in A'$ and $d(x', a_A) < \varepsilon$. If $A' = A$, put $x' = a_A$. Then $\alpha' \in G_A(A') \cap S$.

We conclude that G_A is lower semicontinuous. The same argument works to prove that H_A is lower semicontinuous.

It is easily seen that the maps G_A and H_A have decomposable, closed values. So there exist continuous, single-valued selections g_A and h_A of G_A and H_A , respectively by Theorem 3.1.

Assume that $\text{card}(A) \geq 2$. By the construction,

$$\|g_A(A) - h_A(A)\|_1 = \int_0^1 \|g_A(A)(t) - h_A(A)(t)\| dt = \|a_A - b_A\| > 0.$$

Since the selections g_A and h_A are continuous, there exists an open ball W_A centered at A of radius less than $\frac{1}{2}d_H(A, \exp_1(X))$ in the space $(\exp(X), d_H)$ such that $\|g_A(A') - h_A(A')\|_1 > 0$ for all $A' \in W_A$.

The open cover $\mathcal{W} = \{W_A \mid A \in \exp(X) \setminus \exp_1(X)\}$ admits a locally finite open refinement \mathcal{V} of \mathcal{W} which is a cover of $\exp(X) \setminus \exp_1(X)$. For each $V \in \mathcal{V}$ choose $A_V \in \exp(X) \setminus \exp_1(X)$ where $V \subset W_{A_V}$. Let $\{\lambda_V: \exp(X) \setminus \exp_1(X) \rightarrow [0, 1]\}_{V \in \mathcal{V}}$ be a partition of unity, subordinate to the cover \mathcal{V} . Consider any linear ordering \leq of the cover \mathcal{V} . We now define the desired functions $g, h: \exp(X) \rightarrow L_1(I, X)$ as follows: for $B \in \exp(X) \setminus \exp_1(X)$ and $t \in [0, 1]$ such that

$$\sum_{V < U} \lambda_V(B) \leq t < \sum_{V \leq U} \lambda_V(B)$$

where $B \in U \in \mathcal{V}$ with $\lambda_U(B) > 0$ let

$$g(B)(t) = g_{A_U}(B) \left(\frac{1}{\lambda_U(B)} \left(t - \sum_{V < U} \lambda_V(B) \right) \right)$$

and

$$h(B)(t) = h_{A_U}(B) \left(\frac{1}{\lambda_U(B)} \left(t - \sum_{V < U} \lambda_V(B) \right) \right).$$

If $\text{card}(B) = 1$ we let $g(B) = h(B) = \{a_B\} = \{b_B\} = L_1(I, B)$.

In the case of $\text{card}(B) \geq 2$ we can rewrite the above definitions of $g(B)$ and $h(B)$ in terms of the transformation described in Proposition 3.2. Let us use the denotations

$$\tau_U(B) = \sum_{V < U} \lambda_V(B) \text{ and } \theta_U(B) = \sum_{V \leq U} \lambda_V(B).$$

Then for each $t \in [0, 1]$,

$$g(B)(t) = \sum_{V \in \mathcal{V}} (g_{A_V}(B))^{\tau_V(B), \theta_V(B)}(t)$$

and

$$h(B)(t) = \sum_{V \in \mathcal{V}} (h_{A_V}(B))^{\tau_V(B), \theta_V(B)}(t).$$

Now we are going to prove the continuity of g and h . We will provide the proof for g and the same argument will work for h . Let $B \in \exp(X)$ and let $\varepsilon > 0$ be fixed. We will find a neighbourhood $O(B)$ of B in $\exp(X)$ such that $\int_0^1 \|g(B)(t) - g(B')(t)\| dt < \varepsilon$ for each $B' \in O(B)$. We have the following two cases:

Case 1. $\text{card}(B) \geq 2$. Let $\{V_1, \dots, V_k\} \subset \mathcal{V}$ be the set of all the elements of the cover \mathcal{V} such that $B \in V_m$ for each $m \in \{1, \dots, k\}$. Then for each $t \in [0, 1]$,

$$g(B)(t) = \sum_{m=1}^k (g_{A_{V_m}}(B))^{\tau_{V_m}(B), \theta_{V_m}(B)}(t).$$

There exists an open ball $O(B)$ in $\exp(X) \setminus \exp_1(X)$ centered at B of radius η small enough so that the following conditions are satisfied:

- i) $O(B)$ meets only finitely many elements of the cover \mathcal{V} . Let $\{V_1, \dots, V_k\} \cup \{V'_1, \dots, V'_{k'}\}$ be the set of distinct elements of \mathcal{V} which meet $O(B)$;
- ii) $\eta < \min\{d_H(B, \exp(X) \setminus V_m) \mid m \in \{1, \dots, k\}\}$ (Note that this implies $B' \in V_1 \cap \dots \cap V_k$ for each $B' \in O(B)$ and so $\text{card}(B') \geq 2$);
- iii) if $B' \in O(B)$ then

$$\sum_{j=1}^{k'} \lambda_{V'_j}(B') < \frac{\varepsilon}{3};$$

- iv)

$$\left\| (g_{A_{V_m}}(B))^{\tau_{V_m}(B), \theta_{V_m}(B)} - (g_{A_{V_m}}(B))^{\tau_{V_m}(B'), \theta_{V_m}(B')} \right\|_1 < \frac{\varepsilon}{3k}$$

for each $B' \in O(B)$ and $m \in \{1, \dots, k\}$ (This is possible by the continuity of the transformation from Proposition 3.2 applied to the function $g_{A_{V_m}}(B)$ and the subinterval $[\tau_{V_m}(B), \theta_{V_m}(B)]$ of $[0, 1]$ for each $m \in \{1, \dots, k\}$.);

- v)

$$\|g_{A_{V_m}}(B) - g_{A_{V_m}}(B')\|_1 < \frac{\varepsilon}{3k}$$

for each $B' \in O(B)$ and $m \in \{1, \dots, k\}$ (This is possible by the continuity of each $g_{A_{V_m}}$ at B , $m \in \{1, \dots, k\}$.)

Now we choose any $B' \in O(B)$ and prove that $\|g(B) - g(B')\|_1 < \varepsilon$. We rewrite the definition of $g(B')$ as we did for $g(B)$ using the transformation described in Proposition 3.2 (note that $\text{card}(B') \geq 2$ by ii). For each $t \in [0, 1]$ we obtain

$$g(B')(t) = \sum_{m=1}^k (g_{A_{V_m}}(B'))^{\tau_{V_m}(B'), \theta_{V_m}(B')}(t) + \sum_{j=1}^{k'} (g_{A_{V'_j}}(B'))^{\tau_{V'_j}(B'), \theta_{V'_j}(B')}(t).$$

By our initial assumption, $d(x, y) \leq 1$ for all $x, y \in X$ and since (X, d) is embedded in the Banach space E isometrically, $\text{diam}_{\|\cdot\|}(X) \leq 1$ where $\text{diam}_{\|\cdot\|}(X)$ is the diameter of X as a subset of E with respect to the norm on E . Using condition iii) we obtain

$$\sum_{j=1}^{k'} \left\| (g_{A_{V'_j}}(B'))^{\tau_{V'_j}(B'), \theta_{V'_j}(B')} \right\|_1 \leq \sum_{j=1}^{k'} \lambda_{V'_j}(B') < \frac{\varepsilon}{3}.$$

Then

$$\begin{aligned} \|g(B) - g(B')\|_1 &\leq \sum_{m=1}^k \left\| (g_{A_{V_m}}(B))^{\tau_{V_m}(B), \theta_{V_m}(B)} - (g_{A_{V_m}}(B'))^{\tau_{V_m}(B'), \theta_{V_m}(B')} \right\|_1 + \\ &\quad \sum_{j=1}^{k'} \left\| (g_{A_{V'_j}}(B'))^{\tau_{V'_j}(B'), \theta_{V'_j}(B')} \right\|_1 \leq \\ &\quad \sum_{m=1}^k \left\| (g_{A_{V_m}}(B))^{\tau_{V_m}(B), \theta_{V_m}(B)} - (g_{A_{V_m}}(B))^{\tau_{V_m}(B'), \theta_{V_m}(B')} \right\|_1 + \\ &\quad \sum_{m=1}^k \left\| (g_{A_{V_m}}(B))^{\tau_{V_m}(B'), \theta_{V_m}(B')} - (g_{A_{V_m}}(B'))^{\tau_{V_m}(B'), \theta_{V_m}(B')} \right\|_1 + \frac{\varepsilon}{3} < \\ &\quad \sum_{m=1}^k \frac{\varepsilon}{3k} + \sum_{m=1}^k \frac{\varepsilon}{3k} + \frac{\varepsilon}{3} = \varepsilon \text{ (by iv) and v)).} \end{aligned}$$

Case 2. $\text{card}(B) = 1$. Then B is the singleton $\{a_B\} = \{b_B\}$ and by the assumption, $g(B) = \{a_B\}$, the constant function. Consider the open ball $O(B)$ in $\exp(X)$ centered at B of radius ε .

If $B' \in O(B)$ with $\text{card}(B') = 1$, we obtain $\|g(B) - g(B')\|_1 = \|a_B - a_{B'}\| < \varepsilon$.

Now assume that $B' \in O(B)$ and $\text{card}(B') \geq 2$. Let $\{V_1, \dots, V_k\}$ be the set of all the elements of the cover \mathcal{V} such that $B' \in V_m$ for each $m \in \{1, \dots, k\}$. Then for every $t \in [0, 1]$ and $m \in \{1, \dots, k\}$, $g_{A_{V_m}}(B')(t) \in B'$ and so

$$\left\| a_B - (g_{A_{V_m}}(B'))^{\tau_{V_m}(B'), \theta_{V_m}(B')}(t) \right\| < \varepsilon.$$

We obtain

$$\|g(B) - g(B')\|_1 < \sum_{m=1}^k \lambda_{V_m}(B') \varepsilon = \varepsilon.$$

The above argument works for the function h as well. We conclude that the functions $g, h: \exp(X) \rightarrow L_1(I, X)$ are both continuous and satisfy the condition $\|g(B) - h(B)\|_1 > 0$ for every $B \in \exp X \setminus \exp_1(X)$. It is clear from the construction that $g(B) = h(B)$ for each $B \in \exp_1(X)$. \square

An extension operator T satisfying the properties listed in Theorem 2.1 will be constructed as the sum of the series $\sum_{n=1}^{\infty} \frac{1}{2^n} T_n$ with operators $T_n: C^\dagger \rightarrow C^*(X \times X)$ to be defined below.

For each $n \in \mathbb{N}$ let \mathcal{U}_n be a locally finite open cover of the space X such that $\text{diam}_d(U) < 2^{-n}$ for every $U \in \mathcal{U}_n$. Fix a partition of unity $\{\lambda_U^n: X \rightarrow [0, 1]\}_{U \in \mathcal{U}_n}$ subordinate to the cover \mathcal{U}_n . Give \mathcal{U}_n the discrete topology. Let \leq_n be any linear ordering on \mathcal{U}_n .

Let $\mathcal{U} = \bigsqcup_{n=1}^{\infty} \mathcal{U}_n$ be the disjoint union of the discrete topological spaces \mathcal{U}_n , $n \in \mathbb{N}$ and let $X \sqcup \mathcal{U}$ be the disjoint union of X and \mathcal{U} . Recall that the space X is isometrically embedded in the Banach space E as a closed, bounded subspace. We may assume that in fact $X \sqcup \mathcal{U}$ is embedded in E as a closed, bounded subspace. We identify $L_1(I, A)$ for $A \in \exp X$ and each $L_1(I, \mathcal{U}_n)$ with closed subsets of $L_1(I, X \sqcup \mathcal{U})$.

For each $n \in \mathbb{N}$ let $f_n: X \rightarrow L_1(I, \mathcal{U}_n)$ be defined as follows:

$$f_n(x)(t) = U \text{ if } \sum_{V <_n U} \lambda_V^n(x) \leq_n t < \sum_{V \leq_n U} \lambda_V^n(x) \text{ for } x \in X.$$

It is easily checked as in Proposition 4.2 that the map f_n is continuous.

Now for each $n \in \mathbb{N}$ define a map $r_n: X \times \exp(X) \rightarrow L_1(I, X \sqcup \mathcal{U})$ by the formula

$$r_n(x, A)(t) = \begin{cases} f_n(x)(t) & \text{if } 0 \leq t < \min\{1, nd(x, A)\}; \\ f(x, A)(t) & \text{if } \min\{1, nd(x, A)\} \leq t \leq 1 \end{cases}$$

for every $(x, A) \in X \times \exp(X)$. Here f is the selection defined following the proof of Proposition 4.1. We note that for each $n \in \mathbb{N}$, r_n is a continuous function. The proof of this fact is similar to the one provided for the functions g and h (see Proposition 3.2 and Proposition 4.2).

Using the functions g and h from Proposition 4.2 we define the linear operator $D: C^\dagger \rightarrow \bigcup_{A \in \exp(X)} \mathbb{R}^{(A \sqcup \mathcal{U}) \times (A \sqcup \mathcal{U})}$ as follows. For $p \in C^\dagger$ let

$$D(p)(x, y) = \begin{cases} p(x, y), & \text{if } x, y \in \text{dom } p; \\ \frac{1}{2} \int_0^1 p(x(s), g(\text{dom } p)(s)) ds + \frac{1}{2} \int_0^1 p(x(s), h(\text{dom } p)(s)) ds, & \text{if } x \in \text{dom } p \text{ and } y \in \mathcal{U}; \\ \frac{1}{2} \int_0^1 p(g(\text{dom } p)(s), y(s)) ds + \frac{1}{2} \int_0^1 p(h(\text{dom } p)(s), y(s)) ds, & \text{if } x \in \mathcal{U} \text{ and } y \in \text{dom } p; \\ \int_0^1 p(g(\text{dom } p)(s), h(\text{dom } p)(s)) ds, & \text{if } x, y \in \mathcal{U} \text{ and } x \neq y; \\ 0, & \text{if } x, y \in \mathcal{U} \text{ and } x = y. \end{cases}$$

Define the map $T_n: C^\dagger \rightarrow \mathbb{R}^{X \times X}$ by the formula

$$T_n(p)(x, y) = \int_0^1 D(p)(r_n(x, \text{dom } p)(t), r_n(y, \text{dom } p)(t)) dt$$

for $p \in C^\dagger$ and $x, y \in X$. Finally we define $T: C^\dagger \rightarrow \mathbb{R}^{(X \times X)}$ by $T = \sum_{n=1}^{\infty} \frac{1}{2^n} T_n$.

5. PROPERTIES OF THE OPERATOR T

The propositions of the current section together provide the proof of Theorem 2.1 which is the main result of this paper.

Proposition 5.1. *The map T is an extension operator.*

Proof. Fix any $p \in C^\dagger$ with $\text{dom } p = A$ and any point $(x, y) \in A \times A$. Let us choose an arbitrary $n \in \mathbb{N}$ and consider $T_n(p)(x, y)$. Since $d(x, A) = d(y, A) = 0$ we see that $r_n(x, A) = f(x, A)$ and $r_n(y, A) = f(y, A)$. By the properties of the map f , we obtain $f(x, A) = \{x\} = L_1(I, \{x\})$ and $f(y, A) = \{y\} = L_1(I, \{y\})$ because $x, y \in A$. Therefore, using the definition of the map D we have

$$T_n(p)(x, y) = \int_0^1 D(p)(x(t), y(t))dt = \int_0^1 p(x(t), y(t))dt = p(x, y).$$

We see that each T_n is an extension operator and so

$$T(p)(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} T_n(p)(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} p(x, y) = p(x, y),$$

as required. \square

Proposition 5.2. *The map T is linear.*

Proof. Let $p_1, p_2 \in C^\dagger$ with $\text{dom } p_1 = \text{dom } p_2 = A$ and let $s_1, s_2 \in \mathbb{R}$. Then for each $x, y \in X$ we obtain

$$\begin{aligned} T(s_1 p_1 + s_2 p_2)(x, y) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \int_0^1 D(s_1 p_1 + s_2 p_2)(r_n(x, A)(t), r_n(y, A)(t))dt = \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \int_0^1 [s_1 D(p_1) + s_2 D(p_2)](r_n(x, A)(t), r_n(y, A)(t))dt = \\ &= s_1 T(p_1)(x, y) + s_2 T(p_2)(x, y) \end{aligned}$$

because the map D is linear. \square

Proposition 5.3. *The map T is regular.*

Proof. Let $p \in C^\dagger$ with $\text{dom } p = A$ and $n \in \mathbb{N}$ be arbitrary. First let us show that $\|D(p)\|_\infty = \sup\{|D(p)(x, y)| \mid x, y \in X \sqcup \mathcal{U}\} = \|p\|_\infty$. Indeed $\|D(p)\|_\infty \leq \|p\|_\infty$ because

$$\begin{aligned} \left| \frac{1}{2} \int_0^1 p(x(s), g(A)(s))ds + \frac{1}{2} \int_0^1 p(x(s), h(A)(s))ds \right| &\leq \frac{1}{2} \|p\|_\infty + \frac{1}{2} \|p\|_\infty = \|p\|_\infty, \\ \left| \frac{1}{2} \int_0^1 p(g(A)(s), y(s))ds + \frac{1}{2} \int_0^1 p(h(A)(s), y(s))ds \right| &\leq \frac{1}{2} \|p\|_\infty + \frac{1}{2} \|p\|_\infty = \|p\|_\infty \\ \text{and } \left| \int_0^1 p(g(A)(s), h(A)(s))ds \right| &\leq \|p\|_\infty. \end{aligned}$$

The inequality $\|D(p)\|_\infty \leq \|p\|_\infty$ cannot be strict because D is an extension operator from C^\dagger to $\bigcup_{A \in \text{exp}(X)} \mathbb{R}^{(A \sqcup \mathcal{U}) \times (A \sqcup \mathcal{U})}$. Therefore $\|D(p)\|_\infty = \|p\|_\infty$ and we obtain

$$\|T_n(p)\|_\infty = \sup_{x, y \in X} \left| \int_0^1 D(p)(r_n(x, A)(t), r_n(y, A)(t))dt \right| \leq \|p\|_\infty.$$

Again, since T_n is an extension operator, we obtain the equality $\|T_n(p)\|_\infty = \|p\|_\infty$ for any $n \in \mathbb{N}$. So $\|T(p)\|_\infty = \sum_{n=1}^{\infty} \frac{1}{2^n} \|T_n(p)\|_\infty = \|p\|_\infty$.

We note that $T(p)$ is bounded on $X \times X$ for each $p \in C^\dagger$. \square

Proposition 5.4. *The map T preserves constant functions.*

Proof. If $p \in C^\dagger$ is such that $p(a, b) = \eta \in \mathbb{R}$ for each $a, b \in \text{dom } p$ then $D(p)(x', y') = \eta$ for each $x', y' \in \text{dom } p \sqcup \mathcal{U}$. Therefore, $T(p)(x, y) = \eta$ for all $x, y \in X$. \square

Proposition 5.5. *The operator T preserves pseudometrics.*

Proof. Let $p \in \mathcal{PM}$. From the definition of T it is clear that $T(p)$ is symmetric and $T(p)(x, x) = 0$ for every $x \in X$. The triangle inequality also easily follows from the definition of T . So the map T carries partial, continuous, bounded pseudometrics to bounded pseudometrics on X . We will prove in Proposition 5.10 that the extended pseudometrics are in fact continuous on X . \square

Proposition 5.6. *The operator T preserves metrics.*

Proof. The proof is almost identical to the one in Lemma 4, [3]. To show that T extends a partial metric $p \in \mathcal{M}$ with $\text{dom } p = A$ to a metric on X we need to check that if $x, y \in X$ are distinct points then $T(p)(x, y) > 0$. So let $x, y \in X$ with $x \neq y$ be fixed. If $x, y \in A$, we obtain $T(p)(x, y) = p(x, y) > 0$ because p is a metric on A .

Now assume that $x \in X \setminus A$ and $y \in A$. There exists a number $n \in \mathbb{N}$ such that $d(x, A) > \frac{1}{n}$. So $r_n(x, A) = f_n(x)$ and $r_n(y, A) = \{y\} = L_1(I, \{y\})$. From the definition of the map D we obtain

$$\begin{aligned} D(p)(r_n(x, A)(t), r_n(y, A)(t)) &= \\ \frac{1}{2} \int_0^1 p(g(A)(s), y(s)) ds + \frac{1}{2} \int_0^1 p(h(A)(s), y(s)) ds &\geq \\ \frac{1}{2} \int_0^1 p(g(A)(s), h(A)(s)) ds &> 0 \end{aligned}$$

for every $t \in [0, 1]$ by the properties of the functions g and h . Therefore

$$T_n(p)(x, y) = \int_0^1 D(p)(r_n(x, A)(t), y(t)) dt > 0.$$

If $x \in A$ and $y \in X \setminus A$ we proceed very similarly as above.

Finally we consider the case when $x, y \in X \setminus A$. There exists an $n \in \mathbb{N}$ such that $d(x, A) > \frac{1}{n}$, $d(y, A) > \frac{1}{n}$ and $d(x, y) > \frac{1}{2^{n-1}}$. We obtain $r_n(x, A) = f_n(x)$ and $r_n(y, A) = f_n(y)$. Since $\text{diam}_d(U) < \frac{1}{2^n}$ for $U \in \mathcal{U}_n$ there is no $U \in \mathcal{U}_n$ such that x and y both belong to U . Therefore, by the properties of the functions g and h ,

$$D(p)(f_n(x)(t), f_n(y)(t)) = \int_0^1 p(g(A)(s), h(A)(s)) ds > 0$$

for every $t \in [0, 1]$. We obtain

$$T_n(p)(x, y) = \int_0^1 D(p)(f_n(x)(t), f_n(y)(t)) dt > 0.$$

So $T(p)$ is a metric on X .

We will prove separately in Proposition 5.10 that the extended metrics are continuous on X . \square

Our next auxiliary result is well-known but we include the proof for the completeness of our exposition. It easily follows from properties of convergence of graphs of partial, continuous, bounded functions.

Proposition 5.7. *Let $p \in C^\dagger$ and let $\{p_i\}$ be a sequence from C^\dagger such that $\tilde{d}_H(\Gamma_{p_i}, \Gamma_p) \rightarrow 0$ as $i \rightarrow \infty$. Then if $\{(a_i, b_i)\}$ is a sequence in $X \times X$ with $a_i, b_i \in \text{dom } p_i$ for all $i \in \mathbb{N}$ and converging to (a, b) where $a, b \in \text{dom } p$ then $p_i(a_i, b_i) \rightarrow p(a, b)$ as $i \rightarrow \infty$.*

Proof. Assume that $p_i, p, (a_i, b_i)$ and (a, b) are as specified in the hypothesis. Fix any $\varepsilon > 0$. Since p is continuous at (a, b) , there exists $\delta > 0$ with $\delta \leq \varepsilon$ such that $|p(a, b) - p(a', b')| < \varepsilon/2$ for all $a', b' \in \text{dom } p$ with $\tilde{d}[(a, b), (a', b')] < \delta$. Since $(a_i, b_i) \rightarrow (a, b)$ as $i \rightarrow \infty$ there exists an index $k \in \mathbb{N}$ such that $\tilde{d}[(a, b), (a_i, b_i)] < \delta/2$ whenever $i \geq k$. Fix any $i_0 \geq k$. Since $\tilde{d}_H(\Gamma_{p_i}, \Gamma_p) \rightarrow 0$ as $i \rightarrow \infty$ one can find an index $i_1 \geq i_0$ such that whenever $i \geq i_1$ there exist points $a', b' \in \text{dom } p$ such that

$$\tilde{d}[(a', b', p(a', b')), (a_i, b_i, p_i(a_i, b_i))] < \frac{\delta}{2}.$$

Note that

$$\tilde{d}[(a, b), (a', b')] \leq \tilde{d}[(a, b), (a_i, b_i)] + \tilde{d}[(a_i, b_i), (a', b')] < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Therefore,

$$|p(a, b) - p_i(a_i, b_i)| \leq |p(a, b) - p(a', b')| + |p(a', b') - p_i(a_i, b_i)| < \frac{\varepsilon}{2} + \frac{\delta}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

The following statement is essentially Proposition 4.2 from [16].

Proposition 5.8. *For $p \in C^\dagger$ and a sequence $\{p_i\} \subset C^\dagger$ with $\tilde{d}_H(\Gamma_{p_i}, \Gamma_p) \rightarrow 0$ and any sequences $\{\alpha_i\}, \{\beta_i\} \subset L_1(I, X)$ converging to $\alpha, \beta \in L_1(I, X)$, respectively, one has*

$$\int_0^1 p_i(\alpha_i(t), \beta_i(t)) dt \rightarrow \int_0^1 p(\alpha(t), \beta(t)) dt \text{ as } i \rightarrow \infty.$$

Proof. Let $p_i \rightarrow p$ in C^\dagger , $\alpha_i \rightarrow \alpha$ and $\beta_i \rightarrow \beta$ in $L_1(I, X)$ as $i \rightarrow \infty$. Set $\text{dom } p = A$ and $\text{dom } p_i = A_i$ for each $i \in \mathbb{N}$. Consider the set

$$Y = \left\{ \left(a, b, \frac{1}{i} \right) \mid i \in \mathbb{N}, a, b \in A_i \right\} \cup \{ (a, b, 0) \mid a, b \in A \}.$$

We assume that Y is endowed with the subspace topology when viewed as a subset of the product $X \times X \times \mathbb{R}$ where \mathbb{R} has the standard topology. Define a function $q: Y \rightarrow \mathbb{R}$ by the formula

$$q(a, b, z) = \begin{cases} p(a, b) & \text{if } z = 0; \\ p_i(a, b) & \text{if } z = 1/i \end{cases}$$

for every $(a, b, z) \in Y$.

By Proposition 5.7 if a sequence $\{(a_i, b_i)\}$ in $X \times X$ converges to $(a, b) \in A \times A$ and $(a_i, b_i) \in A_i \times A_i$ for each i then $p_i(a_i, b_i)$ converges to $p(a, b)$ because $\tilde{d}_H(\Gamma_{p_i}, \Gamma_p) \rightarrow 0$ as $i \rightarrow \infty$. Using this condition we conclude that q is a continuous map on Y . For every i let $\gamma_i: I \rightarrow \mathbb{R}$ be the constant function with value $1/i$ and let $\gamma: I \rightarrow \mathbb{R}$ be the constant function with value 0. Then for every i we get

$$\int_0^1 p_i(\alpha_i(t), \beta_i(t)) dt = \int_0^1 q(\alpha_i(t), \beta_i(t), \gamma_i(t)) dt$$

and

$$\int_0^1 p(\alpha(t), \beta(t)) dt = \int_0^1 q(\alpha(t), \beta(t), \gamma(t)) dt.$$

Since $\lim_{i \rightarrow \infty} 1/i = 0$ we see that $\{\gamma_i\}$ converges pointwise to γ on I . Using the definition of the map q and the fact that the sequence $\{p_i\}$ converges to p in C^\dagger , we conclude that the sequence of functions $\{q(\alpha_i, \beta_i, \gamma_i)\}$ is uniformly bounded on I . Suppose that there is a subsequence $(\alpha_{i_m}, \beta_{i_m}, \gamma_{i_m})$ and $\eta \in \mathbb{R}$ such that

$$\int_0^1 q(\alpha_{i_m}(t), \beta_{i_m}(t), \gamma_{i_m}(t)) dt \xrightarrow{i \rightarrow \infty} \eta \neq \int_0^1 q(\alpha(t), \beta(t), \gamma(t)) dt.$$

Since $\|\alpha - \alpha_i\|_1 \rightarrow 0$, $\|\beta - \beta_i\|_1 \rightarrow 0$ and $\{\gamma_i\}$ converges to γ pointwise on I , one can find a subsequence $(\alpha_{i_{m_k}}, \beta_{i_{m_k}}, \gamma_{i_{m_k}})$ that converges to (α, β, γ) almost everywhere on I . Since q is a continuous map we see that the sequence $q(\alpha_{i_{m_k}}, \beta_{i_{m_k}}, \gamma_{i_{m_k}})$ converges to $q(\alpha, \beta, \gamma)$ almost everywhere on I . By Lebesgue's dominated convergence theorem we obtain

$$\int_0^1 q(\alpha_{i_{m_k}}(t), \beta_{i_{m_k}}(t), \gamma_{i_{m_k}}(t)) dt \xrightarrow{i \rightarrow \infty} \int_0^1 q(\alpha(t), \beta(t), \gamma(t)) dt,$$

a contradiction. Therefore,

$$\int_0^1 p_i(\alpha_i(t), \beta_i(t)) dt \rightarrow \int_0^1 p(\alpha(t), \beta(t)) dt \text{ as } i \rightarrow \infty.$$

This completes the proof. \square

Remark. To prove property 8) in the statement of Theorem 2.1 we assume that the set $C^*(X \times X)$ is endowed with the topology of uniform convergence on compact sets. The type of convergence discussed in Proposition 5.7 is called continuous convergence. In the fixed domain case continuous convergence is equivalent to the uniform convergence on compact sets in $C^*(X \times X)$ (see [15, page 109]). So in order to prove the continuity of the map $T: C^\dagger \rightarrow C^*(X \times X)$ we are going to prove the following result.

Proposition 5.9. *Let $p \in C^\dagger$ and let $\{p_i\}$ be a sequence from C^\dagger such that $\tilde{d}_H(\Gamma_{p_i}, \Gamma_p) \rightarrow 0$ as $i \rightarrow \infty$. If $\{(x_i, y_i)\} \subset X \times X$ is a sequence converging to $(x, y) \in X \times X$, then the numerical sequence $T(p_i)(x_i, y_i)$ converges to $T(p)(x, y)$ as $i \rightarrow \infty$.*

Proof. Let $\{p_i\}$ be a sequence from C^\dagger such that $\tilde{d}_H(\Gamma_{p_i}, \Gamma_p) \rightarrow 0$ as $i \rightarrow \infty$ for some $p \in C^\dagger$. Set $\text{dom } p = A$ and $\text{dom } p_i = A_i$ for all $i \in \mathbb{N}$. By Proposition 5.7 the convergence of a sequence $(c_i, e_i) \in A_i \times A_i$ to $(c, e) \in A \times A$ implies that $p_i(c_i, e_i)$ converges to $p(c, e)$ as $i \rightarrow \infty$. Recall that the symbol $c = c(s)$, $s \in I$ means the identification of the point $c \in X \subset E$ with the constant function $c = \{c\}$ in $L_1(I, E)$ with the value c . Similarly for e, c_i and e_i for each $i \in \mathbb{N}$.

Let $g, h: \exp(X) \rightarrow L_1(I, X)$ be the functions defined in Proposition 4.2. The continuity of g and h together with Proposition 5.8 imply the following conditions:

- (1) $\int_0^1 p_i(c_i(s), g(A_i)(s)) ds \rightarrow \int_0^1 p(c(s), g(A)(s)) ds$ as $i \rightarrow \infty$,
- (2) $\int_0^1 p_i(c_i(s), h(A_i)(s)) ds \rightarrow \int_0^1 p(c(s), h(A)(s)) ds$ as $i \rightarrow \infty$,

$$(3) \quad \int_0^1 p_i(g(A_i)(s), e_i(s)) ds \rightarrow \int_0^1 p(g(A)(s), e(s)) ds \text{ as } i \rightarrow \infty,$$

$$(4) \quad \int_0^1 p_i(h(A_i)(s), e_i(s)) ds \rightarrow \int_0^1 p(h(A)(s), e(s)) ds \text{ as } i \rightarrow \infty,$$

$$(5) \quad \int_0^1 p_i(g(A_i)(s), h(A_i)(s)) ds \rightarrow \int_0^1 p(g(A)(s), h(A)(s)) ds \text{ as } i \rightarrow \infty.$$

At the end of Section 4 we defined operators T_n , $n \in \mathbb{N}$ and $T = \sum_{n=1}^{\infty} \frac{1}{2^n} T_n$. Fix any $n \in \mathbb{N}$ and consider the operator T_n . The formula in the definition of T_n is somewhat similar to that in the statement of Proposition 5.8. The difference is that we need to include the operator D in the formula for T_n . We will proceed by modifying the proof of Proposition 5.8. Define

$$Y = \left\{ \left(a, b, \frac{1}{i} \right) \mid i \in \mathbb{N} \text{ and } a, b \in A_i \sqcup \mathcal{U} \right\} \cup \{ (a, b, 0) \mid a, b \in A \sqcup \mathcal{U} \}$$

as a subspace of the space $(X \sqcup \mathcal{U}) \times (X \sqcup \mathcal{U}) \times \mathbb{R}$. Define a map $q: Y \rightarrow \mathbb{R}$ by the formula

$$q(a, b, z) = \begin{cases} D(p)(a, b) & \text{if } z = 0; \\ D(p_i)(a, b) & \text{if } z = \frac{1}{i} \end{cases}$$

for every $(a, b, z) \in Y$.

We prove that the map q defined above is continuous on Y . Take any convergent sequence $\{(a_j, b_j, z_j)\}$ in Y and consider the following cases.

Case 1. $z = \lim_{j \rightarrow \infty} z_j = \frac{1}{i}$ for some $i \in \mathbb{N}$. Then eventually $z_j = \frac{1}{i}$ and so eventually $a_j, b_j \in A_i \sqcup \mathcal{U}$ and the limits $a = \lim_{j \rightarrow \infty} a_j$ and $b = \lim_{j \rightarrow \infty} b_j$ both belong to $A_i \sqcup \mathcal{U}$. We see that for large j , $q(a_j, b_j, z_j) = D(p_i)(a_j, b_j)$.

We have the following subcases:

- i) If $a \in X$ and $b \in \mathcal{U}$ then for large j , $a_j \in A_i$, $b_j = b \in \mathcal{U}$ and $p_j = p_i$. Use the definition of D and conditions (1) and (2) of this proof to get that $D(p_j)(a_j, b_j)$ converges to $D(p_i)(a, b)$ as $j \rightarrow \infty$.
- ii) If $a \in \mathcal{U}$ and $b \in X$ then for large j , $a_j = a \in \mathcal{U}$, $b_j \in A_i$ and $p_j = p_i$. Using the definition of D together with conditions (3) and (4) of this proof we see that $D(p_j)(a_j, b_j)$ converges to $D(p_i)(a, b)$ as $j \rightarrow \infty$.
- iii) If $a, b \in \mathcal{U}$ then for large j , $a_j = a \in \mathcal{U}$, $b_j = b \in \mathcal{U}$ and $p_j = p_i$. Use the definition of D together with condition (5) of this proof again to get that $D(p_j)(a_j, b_j)$ converges to $D(p_i)(a, b)$ as $j \rightarrow \infty$.
- iv) Finally if $a, b \in X$ then $a_j, b_j \in A_i$ for large j and so $a, b \in A_i$ since A_i is closed in X . We use the properties of D and the continuity of p_i to obtain

$$D(p_i)(a_j, b_j) = p_i(a_j, b_j) \rightarrow p_i(a, b) = D(p_i)(a, b) \text{ as } j \rightarrow \infty.$$

So for large j we have

$$q(a_j, b_j, z_j) = D(p_i)(a_j, b_j) \rightarrow D(p_i)(a, b) = q\left(a, b, \frac{1}{i}\right).$$

Case 2. $z = \lim_{j \rightarrow \infty} z_j = 0$. If eventually $z_j = 0$, we would proceed as in Case 1 and use only the map p for large j . If we assume that z_j is not 0 eventually we have to show that $D(p_j)(a_j, b_j)$ converges to $D(p)(a, b)$ provided $(a_j, b_j, z_j) \rightarrow (a, b, 0)$ in

Y . The proof is almost identical to the one provided in Case 1. The only difference is that we would use the maps p_j to define $q(a_j, b_j, z_j)$ and the limiting map p to define $q(a, b, 0)$.

We conclude that the map q is continuous on Y . Note also that the map q is bounded since $\tilde{d}_H(\Gamma_{p_i}, \Gamma_p) \rightarrow 0$ as $i \rightarrow \infty$.

Let $\{(x_i, y_i)\}$ be a sequence in $X \times X$ converging to (x, y) in $X \times X$. Let $r_n: X \times \exp(X) \rightarrow L_1(I, X \sqcup \mathcal{U})$ be as defined near the end of Section 4. Set $r_n(x, A) = \alpha$, $r_n(y, A) = \beta$, $r_n(x_i, A_i) = \alpha_i$ and $r_n(y_i, A_i) = \beta_i$ for each $i \in \mathbb{N}$.

By the continuity of the map r_n , $\alpha_i \rightarrow \alpha$ and $\beta_i \rightarrow \beta$ in $L_1(I, X)$ as $i \rightarrow \infty$. Denote by $\gamma_i, \gamma: I \rightarrow \mathbb{R}$ the constant functions $1/i$ and 0 respectively for each $i \in \mathbb{N}$. We obtain

$$T_n(p_i)(x_i, y_i) = \int_0^1 q(\alpha_i(t), \beta_i(t), \gamma_i(t)) dt \text{ and}$$

$$T_n(p)(x, y) = \int_0^1 q(\alpha(t), \beta(t), \gamma(t)) dt.$$

Since the map q is continuous and bounded we use a similar argument as in the proof of Proposition 5.8 to conclude that

$$(*) \quad T_n(p_i)(x_i, y_i) \rightarrow T_n(p)(x, y) \text{ as } i \rightarrow \infty.$$

Recall that for each $n \in \mathbb{N}$ the map T_n is regular and so $\|T_n(p_i)\|_\infty = \|p_i\|_\infty$ for all $n, i \in \mathbb{N}$ and $\|T(p)\|_\infty = \|p\|_\infty$. Since all the maps p_i and p are uniformly bounded, so are $T_n(p_i)$ and $T(p)$, $i \in \mathbb{N}$.

Then for every $\varepsilon > 0$ there exist $n_0, i_0 \in \mathbb{N}$ such that

$$|T_n(p)(x, y) - T_n(p_i)(x_i, y_i)| < \frac{\varepsilon}{2} \text{ for each } n \in \{1, \dots, n_0\} \text{ and } i > i_0$$

by condition $(*)$ and

$$\frac{1}{2^n} |T_n(p)(x, y) - T_n(p_i)(x_i, y_i)| < \frac{\varepsilon}{2} \text{ for all } n > n_0$$

by the uniform boundedness of the maps $T_n(p_i)$ and $T(p)$. This means that for $i > i_0$,

$$|T(p)(x, y) - T(p_i)(x_i, y_i)| \leq$$

$$\sum_{n=1}^{n_0} \frac{1}{2^n} |T_n(p)(x, y) - T_n(p_i)(x_i, y_i)| + \sum_{n=n_0+1}^{\infty} \frac{1}{2^n} |T_n(p)(x, y) - T_n(p_i)(x_i, y_i)| < \varepsilon.$$

and the proof is complete. \square

Proposition 5.10. *For any function $p \in C^\dagger$ the extended function $T(p)$ is continuous on $X \times X$.*

Proof. Take $p_i = p$ for all $i \in \mathbb{N}$ in Proposition 5.9 and the result follows. \square

Proposition 5.11. *For any fixed $A \in \exp X$ the restriction of the operator T onto $C^*(A \times A)$ is continuous with respect to the pointwise convergence topology on $C^*(A \times A)$ and $C^*(X \times X)$.*

Proof. Take $A_i = A$ for all $i \in \mathbb{N}$ in Proposition 5.9 and the result follows. \square

Proposition 5.12. *For any fixed $A \in \exp X$ the restriction of the operator T onto $C^*(A \times A)$ is continuous with respect to the topology of uniform convergence on compact sets on $C^*(A \times A)$ and $C^*(X \times X)$.*

Proof. Let $A \in \exp(X)$ and suppose that $\{p_i\}$ is a sequence in $C^*(A \times A)$ converging to $p \in C^*(A \times A)$ uniformly on compact sets of $A \times A$. By the remark before Proposition 5.9 it is equivalent to write that for every sequence $(a_i, b_i) \in A \times A$ converging to $(a, b) \in A \times A$ the sequence $p_i(a_i, b_i)$ converges to $p(a, b)$. By Proposition 5.9 if $(x_i, y_i) \in X \times X$, $i \in \mathbb{N}$, converges to $(x, y) \in X \times X$ then $T(p_i)(x_i, y_i) \rightarrow T(p)(x, y)$ as $i \rightarrow \infty$. So the sequence $T(p_i)$ converges to $T(p)$ uniformly on compact sets of $X \times X$. \square

Proposition 5.13. *For any fixed $A \in \exp X$ the restriction of the operator T onto $C^*(A \times A)$ is continuous with respect to the uniform convergence topology on $C^*(A \times A)$ and $C^*(X \times X)$.*

Proof. Assume that $A \in \exp X$ is fixed and let $\{p_i\}$ be a sequence in $C^*(A \times A)$ that converges uniformly to $p \in C^*(A \times A)$ on $A \times A$. Since T is linear and regular we obtain

$$\|T(p) - T(p_i)\|_\infty = \|T(p - p_i)\|_\infty = \|p - p_i\|_\infty \rightarrow 0 \text{ as } i \rightarrow \infty$$

and so $\{T(p_i)\}$ converges to $T(p)$ uniformly on $X \times X$. \square

Proposition 5.14. *The operator T preserves admissible metrics.*

Proof. Let p be an admissible partial metric with $\text{dom } p = A$. Then p is both continuous and dominating on A . By Proposition 5.10 $T(p)$ is continuous on $X \times X$. To prove that $T(p)$ is admissible it is enough to verify that $T(p)$ dominates the topology of X . The proof will be very similar to the one in Lemma 5 from [3]. To show that the metric $T(p)$ is dominating we will prove that for every $x \in X$ and every $\varepsilon \in (0, 1]$ there exists a $\delta > 0$ such that $T(p)(x, x') \geq \delta$ for every $x' \in X$ with $d(x', x) > \varepsilon$.

Let $x \in X$ and $\varepsilon \in (0, 1]$ be fixed. Let $x' \in X$ with $d(x, x') > \varepsilon$. We will consider several cases.

Case 1. Assume that $x \notin A$. There exists an $n \in \mathbb{N}$ such that $d(x, A) > 1/n$ and $2^{-n+1} < \varepsilon$. Let

$$\delta = \frac{1}{2^{n+1}} \int_0^1 p(g(A)(s), h(A)(s)) ds$$

and let $x' \in X$ be any point with $d(x, x') > \varepsilon$. So $d(x, x') > 2^{-n+1}$ and by the properties of the cover \mathcal{U}_n and the map f_n , the sets of values of the function $f_n(x)$ and the function $f_n(x')$ have no points in common. This means that

$$D(p)(f_n(x)(t), f_n(x')(t)) = \int_0^1 p(g(A)(s), h(A)(s)) ds > 0 \text{ for every } t \in [0, 1].$$

Also note that

$$\begin{aligned} D(p)(f_n(x)(t), f(x', A)(t)) &= \\ \frac{1}{2} \int_0^1 p(g(A)(s), f(x', A)(t)(s)) ds &+ \frac{1}{2} \int_0^1 p(h(A)(s), f(x', A)(t)(s)) ds \geq \\ \frac{1}{2} \int_0^1 p(g(A)(s), h(A)(s)) ds &> 0 \end{aligned}$$

where $f(x', A)(t)(s)$ is the constant function with the value $f(x', A)(t)$ for all $s \in [0, 1]$.

We obtain

$$\begin{aligned} 2^n T(p)(x, x') &\geq T_n(p)(x, x') = \int_0^1 D(p)(f_n(x)(t), r_n(x')(t)) dt \geq \\ &\frac{1}{2} \int_0^1 p(g(A)(s), h(A)(s)) ds = 2^n \delta. \end{aligned}$$

Case 2. Assume that $x \in A$. Let $n \in \mathbb{N}$ be large enough so that $2^{-n+1} < \varepsilon$. Since the metric p is dominating, there exists a number $\eta > 0$ such that $p(x, y) > \eta$ for every $y \in A$ with $d(x, y) > \varepsilon/2$. Let

$$\delta = \min \left\{ \frac{1}{2^{n+1}} \int_0^1 p(g(A)(s), h(A)(s)) ds, \frac{1}{2^{n+3}} n \varepsilon \int_0^1 p(g(A)(s), h(A)(s)) ds, \frac{3\eta}{8} \right\}.$$

We fix any point $x' \in X$ such that $d(x, x') > \varepsilon$ and consider two subcases.

(i) Let $d(x', A) \geq \varepsilon/4$. We obtain

$$D(p)(x, f_n(x')(t)) \geq \frac{1}{2} \int_0^1 p(g(A)(s), h(A)(s)) ds > 0$$

for every $t \in [0, 1]$. Therefore,

$$\begin{aligned} 2^n T(p)(x, x') &\geq T_n(p)(x, x') = \int_0^1 D(p)(x, r_n(x')(t)) dt \geq \\ &\int_0^{\min\{1, nd(x', A)\}} D(p)(x, f_n(x')(t)) dt \geq \\ &\min\{1, nd(x', A)\} \frac{1}{2} \int_0^1 p(g(A)(s), h(A)(s)) ds > \\ &\frac{1}{2} \min\{1, n\varepsilon/4\} \int_0^1 p(g(A)(s), h(A)(s)) ds \geq 2^n \delta. \end{aligned}$$

(ii) Now assume that $d(x', A) < \varepsilon/4$. From the properties of the selection f constructed in Section 4 we see that for each $t \in [0, 1]$, $f(x', A)(t) \in A$ and in a neighbourhood of x' in X of diameter not greater than $\varepsilon/2$. Since $d(x, x') > \varepsilon$, we have $d(x, f(x')(t)) > \varepsilon/2$ for each $t \in [0, 1]$. By the choice of η we obtain that $p(x, f(x')(t)) > \eta$ for every $t \in [0, 1]$. Therefore,

$$\begin{aligned} 2T(p)(x, x') &\geq T_1(p)(x, x') = \int_0^1 D(p)(x, r_1(x')(t)) dt \geq \\ &\int_{d(x', A)}^1 D(p)(x, f(x')(t)) dt \geq \int_{\varepsilon/4}^1 p(x, f(x')(t)) dt \geq (1 - \frac{\varepsilon}{4})\eta \geq \frac{3}{4}\eta \geq 2\delta. \end{aligned}$$

□

Remark. From Proposition 5.11, 5.12 and 5.13 we obtain continuity in each of the three topologies of pointwise convergence, uniform convergence on compact sets and uniform convergence for the restricted operators $T|_{\mathcal{P}\mathcal{M}(A)}$ for any $A \in \exp(X)$ and $T|_{\mathcal{M}(A)}$ together with $T|_{\mathcal{M}_a(A)}$ for any $A \in \exp(X)$ such that $\text{card}(A) \geq 2$.

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