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EXTENDING PAIRS OF METRICS

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We consider the problem of extension of pairs of continuous and bounded, partial metrics which agree on the non-empty intersections of their domains which are closed and bounded subsets of an arbitrary but fixed metric space. Two pairs of such metrics are close if their corresponding graphs are close and if the intersections of their domains are close in the Hausdorff metric. If, besides, these metrics are uniformly continuous on the intersections of their domains then there is a continuous positive homogeneous operator extending each such a pair of partial metrics to a continuous metric on the union of their domains. We prove that, in general, there is no subadditive extension operator (continuous or not) for such pairs of metrics. We provide examples showing to what extent our results are sharp and we obtain analogous results for ultrametrics.

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Рассматривается задача продолжения пар непрерывных и ограниченных частичных метрик, согласованных на непустых пересечениях их областей определения, которые являются замкнутыми и ограниченными подмножествами произвольного, но фиксированного метрического пространства. Две пары таких метрик близки, если их соответствующие графики и пересечения их областей определения близки относительно метрики Хаусдорфа. Если, кроме того, эти метрики равномерно непрерывны на пересечениях их областей определения, то существует непрерывный положительно однородный оператор, продолжающий каждую пару таких частичных метрик до непрерывной метрики на объединении их областей определения. Доказано, что, вообще говоря, не существует субаддитивного оператора продолжения (непрерывного или нет) для таких пар метрик. Аналогичные результаты получены для ультраметрик.

1. Introduction. The problem of extending a metric from a closed subset of a metrizable topological space to a metric generating the topology of the whole space was initially considered and solved by F. Hausdorff [3]. His result obtained new proofs and was improved in the works of R. Bing, R. Arens, H. Torunczyk and other authors. A counterpart of the Dugundji extension theorem for the case of metrics was obtained by T. Banakh [1]. The next step in the generalization of known results on this topic was related to the problem of simultaneous linear extension of metrics with variale domains. E. D. Tymchatyn and M. Zarichnyi [7] recently constructed a continuous linear operator extending metrics defined on variable closed subsets of a compact metrizable space. In [8] and [6] a similar problem was considered for the case of ultrametrics.

Let A and B be closed subsets of a metric space X such that $A \cap B \neq \varnothing$. R. Bing [4] proved that if ρ_1 and ρ_2 are continuous, partial metrics on A and B respectively which agree

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on $A \cap B$ then one can extend ρ_1 and ρ_2 to a continuous metric on $A \cup B$. We call such a pair of metrics (ρ_1, ρ_2) admissible if additionally ρ_1 and ρ_2 have bounded graphs.

In this paper we consider the problem of simultaneous, continuous extension of all admissible pairs of partial metrics to the unions of their domains. We identify every bounded metric which has a bounded domain with its graph. Two admissible pairs of partial metrics are close if the corresponding graphs are close and the intersections of their domains are close in the appropriate Hausdorff metric.

We use Bing's extension to obtain a continuous extension operator on the set of all admissible pairs of metrics that are uniformly continuous on the intersections of their domains. We give an example which shows that the assumption of uniform continuity of metrics on the intersections of their domains is essential for continuity of the extension operator. We prove that, in general, one cannot extend admissible pairs of uniformly continuous metrics to uniformly continuous metrics. We also provide an example which shows that, in general, one cannot get a counterpart of the result obtained in [7] for the current setting. That is, there is no linear or even subadditive extension operator for admissible pairs of metrics. This answers in the negative the question from the Lviv topological seminar on the existence of such linear extensions. Analogues of the above problems are also solved for the case of ultrametrics.

2. Preliminaries. Let (X, d) be a metric space and denote by $\mathrm{CL}_b(X)$ the space of its nonempty closed bounded subsets with the Hausdorff metric H generated by d. Recall that this means that

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

for every $A, B \in \mathrm{CL}_b(X)$.

Definition 1. A metric ρ on $A \in \mathrm{CL}_b(X)$ is called *continuous* if $\rho(x_n, x) \to 0$ whenever $d(x_n, x) \to 0$ for a sequence $\{x_n\} \subset A$ and $x \in A$.

Definition 2. A metric ρ on $A \in \mathrm{CL}_b(X)$ is called *uniformly continuous* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in A$ we have $\rho(x, y) < \varepsilon$ whenever $d(x, y) < \delta$.

Definition 3. A metric ρ on $A \in \mathrm{CL}_b(X)$ is called *Lipschitz* if there is $\lambda > 0$ such that $\rho(x, y) \le \lambda d(x, y)$ for all $x, y \in A$.

For $A \in \mathrm{CL}_b(X)$ let $\mathcal{M}(A)$ stand for the set of all continuous bounded metrics on A. For every $A \in \mathrm{CL}_b(X)$ the set $\mathcal{M}(A)$ is a positive cone in the sense that it is closed under the operations of pointwise addition and multiplication by a positive number. For $A \in \mathrm{CL}_b(X)$ we write $\mathrm{dom}\rho = A$ if $\rho \in \mathcal{M}(A)$. Every metric ρ from $\mathcal{M}(A)$ can be identified with its graph $\Gamma_\rho = \{(x, y, \rho(x, y)) \in A \times A \times [0, \infty)\}$ which is a closed and bounded subset of the space $X \times X \times \mathbb{R}$ with metric $\tilde{d}$ defined by the formula $\tilde{d}[(a, b, t), (a', b', t')] = d(a, a') + d(b, b') + |t - t'|$ for $a, b, a', b' \in X$ and $t, t' \in \mathbb{R}$. Let $\tilde{H}$ be the Hausdorff metric on $\mathrm{CL}_b(X \times X \times \mathbb{R})$ generated by $\tilde{d}$. Consider the set $\mathcal{M} = \cup\{\mathcal{M}(A) \mid A \in \mathrm{CL}_b(X),\ |A| \ge 2\}$ of all partial continuous bounded metrics with closed and bounded domains in X. Then $\mathcal{M}$ can be viewed as a subspace of the space $\mathrm{CL}_b(X \times X \times \mathbb{R})$. Therefore, we can take the distance between two metrics in $\mathcal{M}$ to be the Hausdorff distance between their graphs. Let

$$\mathcal{P} = \{(\rho_1, \rho_2) \in \mathcal{M}(A) \times \mathcal{M}(B) \mid\ A, B \in \mathrm{CL}_b(X),\ A \cap B \ne \varnothing,\ |A| \ge 2, |B| \ge 2$$
$$\text{and } \rho_1 = \rho_2 \text{ on } (A \cap B) \times (A \cap B)\}.$$

So, the set $\mathcal{P}$ consists of all pairs of partial continuous bounded metrics which agree on the non-empty intersection of their domains. We will call them *admissible pairs of metrics*. A sequence (ρ_1^n, ρ_2^n) from $\mathcal{P}$ converges to $(\rho_1, \rho_2) \in \mathcal{P}$ if and only if

$$\Gamma_{\rho_1^n} \to \Gamma_{\rho_1},\ \Gamma_{\rho_2^n} \to \Gamma_{\rho_2} \text{ in } \mathrm{CL}_b(X \times X \times \mathbb{R})$$

and $\mathrm{dom}\rho_1^n \cap \mathrm{dom}\rho_2^n \to \mathrm{dom}\rho_1 \cap \mathrm{dom}\rho_2$ in $\mathrm{CL}_b(X)$.

Definition 4. Let $A \in \mathrm{CL}_b(X)$ and let $A_0 \subset A$ be closed in A. We will say that a metric $\rho \in \mathcal{M}(A)$ is *uniformly continuous on* A_0 if for every $\varepsilon > 0$ one can find $\delta > 0$ such that for every $a \in A$ and $a_0 \in A_0$ we have $\rho(a, a_0) < \varepsilon$ whenever $d(a, a_0) < \delta$.

Note that the above condition for ρ is stronger than the condition of uniform continuity of the restriction $\rho|_{A_0 \times A_0}$. Let

$$\mathcal{P}_u = \{(\rho_1, \rho_2) \in \mathcal{P} \mid \rho_1 \text{ and } \rho_2 \text{ are uniformly continuous on } \mathrm{dom}\rho_1 \cap \mathrm{dom}\rho_2\}$$

be the subspace of $\mathcal{P}$ consisting of all admissible pairs of partial metrics that are uniformly continuous on the intersection of their domains. Let $\mathbb{N}$ stand for the set of all positive integers.

3. Extending metrics. We will need the following definitions:

Definition 5. A map $u\colon \mathcal{P} \to \mathcal{M}$ is called an *extension operator* if

$$u(\rho_1, \rho_2) \in \mathcal{M}(\mathrm{dom}\rho_1 \cup \mathrm{dom}\rho_2), \quad u(\rho_1, \rho_2)|_{\mathrm{dom}\rho_1 \times \mathrm{dom}\rho_1} = \rho_1 \text{ and } u(\rho_1, \rho_2)|_{\mathrm{dom}\rho_2 \times \mathrm{dom}\rho_2} = \rho_2$$

for every $(\rho_1, \rho_2) \in \mathcal{P}$.

Definition 6. An extension operator $u\colon \mathcal{P} \to \mathcal{M}$ is *positive homogeneous* if $u(c\rho_1, c\rho_2) = cu(\rho_1, \rho_2)$ for every $(\rho_1, \rho_2) \in \mathcal{P}$ and $c > 0$.

The operator u is called *additive* (respectively, *subadditive*) if

$$u((\rho_1, \rho_2) + (\sigma_1, \sigma_2)) = (\text{ respectively, } \leq)\ u(\rho_1, \rho_2) + u(\sigma_1, \sigma_2)$$

for every $(\rho_1, \rho_2), (\sigma_1, \sigma_2) \in \mathcal{P}$ with $\mathrm{dom}\rho_1 = \mathrm{dom}\sigma_1$ and $\mathrm{dom}\rho_2 = \mathrm{dom}\sigma_2$.

The operator u is called *linear* if it is additive and positive homogeneous.

Theorem 1. *There exists an operator $u\colon \mathcal{P} \to \mathcal{M}$ with the following properties:*

(i) u is an extension operator;

(ii) u is positive homogeneous;

(iii) for every $(\rho_1, \rho_2), (\sigma_1, \sigma_2) \in \mathcal{P}$ with $\mathrm{dom}\rho_1 = \mathrm{dom}\sigma_1$ and $\mathrm{dom}\rho_2 = \mathrm{dom}\sigma_2$ we have $u((\rho_1, \rho_2) + (\sigma_1, \sigma_2)) \geq u(\rho_1, \rho_2) + u(\sigma_1, \sigma_2)$;

(iv) the restriction $u|_{\mathcal{P}_u}$ is a continuous map.

Proof. For $(\rho_1, \rho_2) \in \mathcal{P}$ with $\mathrm{dom}\rho_1 = A$ and $\mathrm{dom}\rho_2 = B$ it is enough to define the distance $u(\rho_1, \rho_2)(x, y)$ between all x and y such that $x \in A \setminus B$ and $y \in B \setminus A$. It is known [4, Theorem 4] that there is an extension of the pair (ρ_1, ρ_2) to a continuous metric $\tilde{\rho}$ on $A \cup B$ defined by the formula $\tilde{\rho}(x, y) = \inf_{a \in A \cap B}\{\rho_1(x, a) + \rho_2(y, a)\}$ for $x \in A \setminus B$ and $y \in B \setminus A$ and $\tilde{\rho}(x, y) = \rho_i(x, y)$ if $x, y \in \mathrm{dom}\rho_i$, $i \in \{1, 2\}$. Let $u(\rho_1, \rho_2) = \tilde{\rho}$ and verify the rest of the conditions stated for u.

It can be easily seen from the definition of the operator u that it is positive homogeneous.

Now suppose that (ρ_1, ρ_2) and (σ_1, σ_2) are pairs of metrics from $\mathcal{P}$ with $\mathrm{dom}\rho_1 = \mathrm{dom}\sigma_1 = A$ and $\mathrm{dom}\rho_2 = \mathrm{dom}\sigma_2 = B$. If $x, y \in A$ we obtain $u(\rho_1 + \sigma_1, \rho_2 + \sigma_2)(x, y) = \rho_1(x, y) + \sigma_1(x, y) = u(\rho_1, \rho_2)(x, y) + u(\sigma_1, \sigma_2)(x, y)$. The case when $x, y \in B$ is similar. Now for every $x \in A \setminus B$, $y \in B \setminus A$ by properties of inf we obtain

$$u(\rho_1 + \sigma_1, \rho_2 + \sigma_2)(x, y) = \inf_{a \in A \cap B} \{\rho_1(x, a) + \sigma_1(x, a) + \rho_2(y, a) + \sigma_2(y, a)\} \geq$$
$$\geq \inf_{a \in A \cap B} \{\rho_1(x, a) + \rho_2(y, a)\} + \inf_{a \in A \cap B} \{\sigma_1(x, a) + \sigma_2(y, a)\} = u(\rho_1, \rho_2)(x, y) + u(\sigma_1, \sigma_2)(x, y).$$

Finally, let us prove the continuity of the restriction $u|_{\mathcal{P}_u}$. Let (ρ_1^n, ρ_2^n) be a sequence in $\mathcal{P}_u$ converging to $(\rho_1, \rho_2) \in \mathcal{P}_u$, $\mathrm{dom}\rho_1^n = A_n$, $\mathrm{dom}\rho_2^n = B_n$, $\mathrm{dom}\rho_1 = A$, $\mathrm{dom}\rho_2 = B$. Note that this implies $H(A_n, A) \to 0$, $H(B_n, B) \to 0$ and $H(A_n \cap B_n) \to H(A \cap B)$ as $n \to \infty$. We are going to prove that $\tilde{H}(\Gamma_{u(\rho_1^n, \rho_2^n)}, \Gamma_{u(\rho_1, \rho_2)}) \to 0$ as $n \to \infty$.

Choose an arbitrary $\varepsilon > 0$. Since ρ_1 and ρ_2 are uniformly continuous on $A \cap B$, there exists $0 < \delta < \varepsilon/4$ such that

(a) for every $x \in A$ and $a \in A \cap B$ we have $\rho_1(x, a) < \varepsilon/8$ whenever $d(x, a) < \delta$;

(b) for every $y \in B$ and $a \in A \cap B$ we have $\rho_2(y, a) < \varepsilon/8$ whenever $d(y, a) < \delta$.

Then for all sufficiently large n the following conditions are satisfied:

(1) $H(A_n \cap B_n, A \cap B) < \delta/4$;

(2) $\tilde{H}(\Gamma_{\rho_1^n}, \Gamma_{\rho_1}) < \delta/4$;

(3) $\tilde{H}(\Gamma_{\rho_2^n}, \Gamma_{\rho_2}) < \delta/4$.

Suppose that n is fixed and large enough so that the above conditions are true. Take any point $(x_n, y_n, u(\rho_1^n, \rho_2^n)(x_n, y_n)) \in \Gamma_{u(\rho_1^n, \rho_2^n)}$. Since u is an extension operator in the case when $x_n, y_n \in A_n$, by (2) there exist $x, y \in A$ such that

$$d(x, x_n) + d(y, y_n) + |u(\rho_1, \rho_2)(x, y) - u(\rho_1^n, \rho_2^n)(x_n, y_n)| =$$
$$= d(x, x_n) + d(y, y_n) + |\rho_1(x, y) - \rho_1^n(x_n, y_n)| < \delta/4 < \varepsilon.$$

If $x_n, y_n \in B_n$ we use (3) to get the needed inequality.

Now suppose that $x_n \in A_n \setminus B_n$ and $y_n \in B_n \setminus A_n$. Since

$$u(\rho_1^n, \rho_2^n)(x_n, y_n) = \inf_{a \in A_n \cap B_n} \{\rho_1^n(x_n, a) + \rho_2^n(y_n, a)\},$$

one can find $b_n \in A_n \cap B_n$ such that $\rho_1^n(x_n, b_n) + \rho_2^n(y_n, b_n) - \varepsilon/8 < u(\rho_1^n, \rho_2^n)(x_n, y_n)$. Using (2) and (3), we find points $(x, b', \rho_1(x, b')) \in \Gamma_{\rho_1}$ and $(y, b'', \rho_2(y, b'')) \in \Gamma_{\rho_2}$ such that $d(x, x_n) + d(b', b_n) + |\rho_1(x, b') - \rho_1^n(x_n, b_n)| < \delta/4$ and $d(y, y_n) + d(b'', b_n) + |\rho_2(y, b'') - \rho_2^n(y_n, b_n)| < \delta/4$. Since $b_n \in A_n \cap B_n$, by 1) we can find $b \in A \cap B$ with $d(b, b_n) < \delta/4$. Then $d(b, b') \leq d(b, b_n) + d(b', b_n) < \delta/4 + \delta/4 = \delta/2$ and $d(b, b'') \leq d(b, b_n) + d(b'', b_n) < \delta/4 + \delta/4 = \delta/2$. Therefore, $\rho_1(b, b') < \varepsilon/8$ by (a) and $\rho_2(b, b'') < \varepsilon/8$ by (b). We obtain

$$u(\rho_1, \rho_2)(x, y) \leq \rho_1(x, b) + \rho_2(y, b) \leq \rho_1(x, b') + \rho_1(b, b') + \rho_2(y, b'') + \rho_2(b, b'') <$$
$$\rho_1(x, b') + \frac{\varepsilon}{8} + \rho_2(y, b'') + \frac{\varepsilon}{8} < \rho_1^n(x_n, b_n) + \frac{\delta}{4} + \rho_2^n(y_n, b_n) + \frac{\delta}{4} + \frac{\varepsilon}{4} <$$
$$u(\rho_1^n, \rho_2^n)(x_n, y_n) + \frac{\varepsilon}{8} + \frac{\delta}{2} + \frac{\varepsilon}{4} < u(\rho_1^n, \rho_2^n)(x_n, y_n) + \frac{\varepsilon}{2}.$$

Similarly we show that $u(\rho_1^n,\rho_2^n)(x_n,y_n) < u(\rho_1,\rho_2)(x,y)+3\varepsilon/4$. To do this we will need only to prove that we can use analogues of conditions (a) and (b) for ρ_1^n and ρ_2^n respectively with parameters $\delta/2$ instead of δ and $\varepsilon/4$ instead of $\varepsilon/8$. Suppose that $x_n' \in A_n$, $a_n' \in A_n \cap B_n$ are such that $d(x_n',a_n') < \delta/2$. Then by (2) there exists $(x',a',\rho_1(x',a')) \in \Gamma_{\rho_1}$ such that $d(x',x_n')+d(a',a_n')+|\rho_1(x',a')-\rho_1^n(x_n',a_n')| < \delta/4$. Since

$$d(x',a') \leq d(x',x_n')+d(x_n',a_n')+d(a_n',a') < \frac{\delta}{4}+\frac{\delta}{2}+\frac{\delta}{4}=\delta,$$

we use (a) to obtain $\rho_1^n(x_n',a_n') < \rho_1(x',a')+\delta/4 < \varepsilon/8+\delta/4 < \varepsilon/4$. Similarly, we get $\rho_2^n(y_n',a_n') < \varepsilon/4$ for every $y_n' \in B_n$, $a_n' \in A_n \cap B_n$ with $d(y_n',a_n') < \delta/2$.

Thus, we obtain

$$\tilde{d}\,[(x_n,y_n,u(\rho_1^n,\rho_2^n)(x_n,y_n)),(x,y,u(\rho_1,\rho_2)(x,y))] < \frac{\delta}{4}+\frac{\delta}{4}+\frac{3\varepsilon}{4} < \frac{\varepsilon}{8}+\frac{3\varepsilon}{4} < \varepsilon.$$

Using the same argument as above we can prove that, for every point from the graph of the metric $u(\rho_1,\rho_2)$, there is a point from the graph of $u(\rho_1^n,\rho_2^n)$ which is ε-close. This means that $\tilde{H}(\Gamma_{u(\rho_1^n,\rho_2^n)},\Gamma_{u(\rho_1,\rho_2)}) < \varepsilon$ and so, the restriction of the operator u to $\mathcal{P}_u$ is continuous with respect to the Hausdorff metric topology on $\mathcal{M}$. □

The following example shows that the condition of uniform continuity of metrics on the intersections of their domains is essential for the convergence of their extensions.

Example 1. Let $X=\{x,y,c\}\cup\{a_i \mid i\in\mathbb{N}\}\cup\{b_i \mid i \in \mathbb{N}\}$ be a discrete space and X^* be its one-point compactification. Let d^* be a metric on X^* and $d=d^*|_{X\times X}$. Let $A=\{x,c\}\cup\{a_i \mid i \in \mathbb{N}\}$ and $B=X\setminus\{x\}$. Therefore, $A\cap B=\{c\}\cup\{a_i \mid i\in\mathbb{N}\}$. Define a metric ρ_1 on A by setting $\rho_1(x,z)=1$ if $z\in A\setminus\{x\}$ and $\rho_1(z,z')=1/2$ if $z\neq z'$ and $z,z'\in A\setminus\{x\}$. Let ρ_2 be the metric on B defined as follows:

$$\rho_2(y,c)=\rho_1(y,a_i)=1,\ \rho_2(y,b_i)=\frac{3}{4} \text{ for } i\in\mathbb{N};\ \rho_2(z,z')=\frac{1}{2} \text{ for all other } z,z'\in B,\ z\neq z'.$$

Then ρ_1 and ρ_2 are continuous metrics that agree on $A\cap B$ and which are not uniformly continuous on $A\cap B$. For every $n\in\mathbb{N}$ we define metrics ρ_1^n and ρ_2^n on A and B respectively by $\rho_1^n=\rho_1$ and

$$\rho_2^n(z,z')=\begin{cases}\frac{3}{4}, & \text{if } \{z,z'\}=\{y,a_n\},\\ 1, & \text{if } \{z,z'\}=\{y,b_n\},\\ \rho_2(z,z'), & \text{otherwise.}\end{cases}$$

So, in order to get ρ_2^n from ρ_2 we interchange the ρ_2-distances between y and a_n and y and b_n. To check that $\Gamma_{\rho_2^n}$ converges to Γ_{ρ_2}, consider any $\varepsilon>0$. Suppose that n is large enough so that $d(a_i,b_j)<\varepsilon$ whenever $i,j\geq n$. Then for $(a_n,y,1)\in\Gamma_{\rho_2}$ we choose $(b_n,y,1)\in\Gamma_{\rho_2^n}$ to get $\tilde{d}[(a_n,y,1),(b_n,y,1)]=d(a_n,b_n)<\varepsilon$. Now for the point $(b_n,y,3/4)\in\Gamma_{\rho_2}$ the point $(a_n,y,3/4)\in\Gamma_{\rho_2^n}$ is ε-close. Since all the remaining points from Γ_{ρ_2} are the same as the remaining points in $\Gamma_{\rho_2^n}$, we conclude that $\tilde{H}(\Gamma_{\rho_2^n},\Gamma_{\rho_2})\to 0$ as $n\to\infty$. Since $\tilde{H}(\Gamma_{\rho_1^n},\Gamma_{\rho_1})=0$ and $\mathrm{dom}\rho_1^n\cap\mathrm{dom}\rho_2^n=\mathrm{dom}\rho_1\cap\mathrm{dom}\rho_2$ for all $n\in\mathbb{N}$, we observe that $(\rho_1^n,\rho_2^n)\to(\rho_1,\rho_2)$ in $\mathcal{P}$. From the definition of the extension operator we obtain $u(\rho_1,\rho_2)(x,y)=2$. Now for every $n\in\mathbb{N}$ we get $u(\rho_1^n,\rho_2^n)(x,y)=\rho_1^n(x,a_n)+\rho_2^n(a_n,y)=1+3/4=7/4$.

One can see now that $\tilde{d}[(x,y,2),\Gamma_{u(\rho_1^n,\rho_2^n)}]\geq 1/4$ for every $n\in\mathbb{N}$. So, the graphs of $u(\rho_1^n,\rho_2^n)$ do not converge to the graph of $u(\rho_1,\rho_2)$ in the Hausdorff metric.

From our next example one can see that, in general, there is no subadditive extension operator for pairs of metrics from $\mathcal{P}$.

Example 2. Suppose that there exist discrete subspaces $A = \{x, a, b\}$ and $B = \{y, a, b\}$ of the metric space (X, d) with $x \neq y$ and consider four pairs of uniformly continuous metrics which agree on $A \cap B = \{a, b\}$. Let (ρ_1, ρ_2), (ρ_1', ρ_2'), (σ_1, σ_2), $(\sigma_1', \sigma_2') \in \mathcal{P}$ be defined as follows:

$$\begin{gathered}
\rho_1(x,a) = 1,\ \rho_1(x,b) = 6,\ \rho_2(y,a) = 1,\ \rho_2(y,b) = 4,\ \rho_1(a,b) = \rho_2(a,b) = 5,\\
\rho_1'(x,a) = 6,\ \rho_1'(x,b) = 1,\ \rho_2'(y,a) = 4,\ \rho_2'(y,b) = 1,\ \rho_1'(a,b) = \rho_2'(a,b) = 5,\\
\sigma_1(x,a) = 1,\ \sigma_1(x,b) = 6,\ \sigma_2(y,a) = 2,\ \sigma_2(y,b) = 3,\ \sigma_1(a,b) = \sigma_2(a,b) = 5,\\
\sigma_1'(x,a) = 6,\ \sigma_1'(x,b) = 1,\ \sigma_2'(y,a) = 3,\ \sigma_2'(y,b) = 2,\ \sigma_1'(a,b) = \sigma_2'(a,b) = 5.
\end{gathered}$$

In order to extend these pairs of metrics, we have to define the distances between x and y only. Suppose that $v\colon \mathcal{P} \to \mathcal{M}$ is an additive extension operator. Since $v(\rho_1, \rho_2)$ satisfies the triangle inequality, we should have

$$\begin{gathered}
2 = \rho_1(x,b) - \rho_2(y,b) = v(\rho_1,\rho_2)(x,b) - v(\rho_1,\rho_2)(y,b) \le v(\rho_1,\rho_2)(x,y) \le\\
\le v(\rho_1,\rho_2)(x,a) + v(\rho_1,\rho_2)(y,a) = \rho_1(x,a) + \rho_2(y,a) = 2.
\end{gathered}$$

So, $v(\rho_1, \rho_2)(x, y) = 2$. Similarly, $v(\rho_1', \rho_2')(x, y) = 2$. Now consider the pair $(\rho_1, \rho_2)+(\rho_1', \rho_2') = (\rho_1 + \rho_1', \rho_2 + \rho_2')$. Adding pointwise, we obtain

$$\begin{gathered}
(\rho_1+\rho_1')(x,a) = 7,\ (\rho_1+\rho_1')(x,b) = 7,\ (\rho_2+\rho_2')(y,a) = 5,\ (\rho_2+\rho_2')(y,b) = 5,\\
(\rho_1+\rho_1')(a,b) = (\rho_2+\rho_2')(a,b) = 10.
\end{gathered}$$

Since v is additive, we obtain $v(\rho_1 + \rho_1', \rho_2 + \rho_2')(x, y) = v(\rho_1, \rho_2)(x, y) + v(\rho_1', \rho_2')(x, y) = 2 + 2 = 4$.

Now for $v(\sigma_1, \sigma_2)$ to satisfy the triangle inequality we should require

$$\begin{gathered}
3 = \sigma_1(x,b) - \sigma_2(y,b) = v(\sigma_1,\sigma_2)(x,b) - v(\sigma_1,\sigma_2)(y,b) \le v(\sigma_1,\sigma_2)(x,y) \le\\
\le v(\sigma_1,\sigma_2)(x,a) + v(\sigma_1,\sigma_2)(y,a) = \sigma_1(x,a) + \sigma_2(y,a) = 3.
\end{gathered}$$

So, $v(\sigma_1, \sigma_2)(x, y) = 3$. Similarly, $v(\sigma_1', \sigma_2')(x, y) = 3$.

If we consider the pair of metrics $(\sigma_1 + \sigma_1', \sigma_2 + \sigma_2')$, then we obtain

$$\begin{gathered}
(\sigma_1+\sigma_1')(x,a) = 7,\ (\sigma_1+\sigma_1')(x,b) = 7,\ (\sigma_2+\sigma_2')(y,a) = 5,\ (\sigma_2+\sigma_2')(y,b) = 5,\\
(\sigma_1+\sigma_1')(a,b) = (\sigma_2+\sigma_2')(a,b) = 10.
\end{gathered}$$

Since v is additive, we obtain

$$v(\sigma_1+\sigma_1', \sigma_2+\sigma_2')(x,y) = v(\sigma_1,\sigma_2)(x,y) + v(\sigma_1',\sigma_2')(x,y) = 3 + 3 = 6.$$

But $(\sigma_1 + \sigma_1', \sigma_2 + \sigma_2') = (\rho_1 + \rho_1', \rho_2 + \rho_2')$, so, we get a contradiction. This means that v cannot be additive and thus cannot be linear. Now using Theorem 1, we can conclude that v cannot be subadditive because the extension described in Theorem 1 is the only possible one for the pairs (ρ_1, ρ_2), (ρ_1', ρ_2'), (σ_1, σ_2) and (σ_1', σ_2').

Now we consider an example which shows that, in general, one cannot extend a pair of uniformly continuous metrics ρ_1 and ρ_2 on $\operatorname{dom}\rho_1$ and $\operatorname{dom}\rho_2$ respectively to a uniformly continuous metric on $\operatorname{dom}\rho_1 \cup \operatorname{dom}\rho_2$.

Example 3. Let (X, d) be the subspace of the real line with the standard metric d defined as follows: $X = \{1,3\} \cup \{2^{-n} \mid n \in \mathbb{N}\} \cup \{3^{-n} \mid n \in \mathbb{N}\}$. Then $A = \{1,3\} \cup \{2^{-n} \mid n \in \mathbb{N}\}$ and $B = \{1,3\} \cup \{3^{-n} \mid n \in \mathbb{N}\}$ are closed subsets of X with $A \cap B = \{1,3\}$. Let σ_1 be the metric on A which coincides with d. Construct a metric σ_2 on B so that the resulting metric space (B, σ_2) is isometric to the subspace $B' = \{1,3\} \cup \{2 + 3^{-n} \mid n \in \mathbb{N}\}$ of the real line with the standard metric d where the isometry $i\colon (B, \sigma_2) \to (B', d)$ is defined as follows: $i(1) = 1$, $i(3) = 3$ and $i(3^{-n}) = 2 + 3^{-n}$ for $n \in \mathbb{N}$. It is clear that (σ_1, σ_2) is an admissible pair of uniformly continuous metrics with respect to the standard metric d on X. We are going to show that there exists a unique extension of the pair (σ_1, σ_2) to a metric $v(\sigma_1, \sigma_2)$ on $A \cup B$ which, however, is not uniformly continuous. Let $a = 1$, $b = 3$, $x_n = 2^{-n}$ and $y_n = 3^{-n}$ for $n \in \mathbb{N}$. We have to define only the distances between x_n and y_k, $n, k \in \mathbb{N}$. We obtain

$$3 - 2^{-n} - (3 - 2 - 3^{-k}) = 2 - 2^{-n} + 3^{-k} = \sigma_1(x_n, b) - \sigma_2(y_k, b) =$$
$$= v(\sigma_1, \sigma_2)(x_n, b) - v(\sigma_1, \sigma_2)(y_k, b) \leq v(\sigma_1, \sigma_2)(x_n, y_k) \leq v(\sigma_1, \sigma_2)(x_n, a) + v(\sigma_1, \sigma_2)(y_k, a) =$$
$$= \sigma_1(x_n, a) + \sigma_2(y_k, a) = 1 - 2^{-n} + 2 + 3^{-k} - 1 = 2 - 2^{-n} + 3^{-k}.$$

So, $v(\sigma_1, \sigma_2)(x_n, y_k) = 2 - 2^{-n} + 3^{-k}$ for every $n, k \in \mathbb{N}$. To see that $v(\sigma_1, \sigma_2)$ is not uniformly continuous on $A \cup B$, we note that $d(x_n, y_k)$ can be made arbitrarily close to zero by taking sufficiently large n and k while $v(\sigma_1, \sigma_2)(x_n, y_k) > 1$ for all $n, k \in \mathbb{N}$.

Observe that σ_1 and σ_2 are also Lipschitz metrics on A and B respectively. So, using Example 2 we see that, in general, there is no linear extension operator preserving Lipschitz property of metrics defined on closed and bounded subsets of X.

4. Extending ultrametrics. As a special case of the above problem, we consider extensions of pairs of ultrametrics which are defined on closed, bounded subsets of a zero-dimensional metric space X and which agree on the intersection of their domains. Recall that a metric r on a set Y is called an ultrametric if it satisfies the strong triangle inequality

$$r(x, y) \leq \max\{r(x, z), r(z, y)\}$$

for every $x, y, z \in Y$. It is known that a metric space Y admits an ultrametric which generates its topology if and only if $\dim Y = 0$. Any triangle in an ultrametric space is isosceles with base length less than or equal to the length of the equal legs. The sum of two ultrametrics need not be an ultrametric, so there is no sense to consider linear operators extending ultrametrics. However, the maximum of two ultrametrics is always an ultrametric. Let X be a zero-dimensional metric space. For $A \in \mathrm{CL}_b(X)$ consider the set $\mathcal{UM}(A)$ of all continuous bounded ultrametrics defined on A. For every $A \in \mathrm{CL}_b(X)$ the set $\mathcal{UM}(A)$ is closed under the operations of taking pointwise maximum and multiplying by a positive number. Let $\mathcal{UM} = \cup\{\mathcal{UM}(A) \mid A \in \mathrm{CL}_b(X),\ |A| \geq 2\}$ be the set of all partial continuous, bounded ultrametrics with domains in $\mathrm{CL}_b(X)$. We may view $\mathcal{UM}$ as a subspace of $\mathcal{M}$, so that it inherits the topology of convergence in the Hausdorff distance in $\mathcal{M}$.

As for the case of metrics, we define the sets of admissible pairs of partial continuous ultrametrics on closed and bounded subsets of X:

$$\mathcal{PU} = \{(\rho_1, \rho_2) \in \mathcal{UM}(A) \times \mathcal{UM}(B) \mid\ A, B \in \mathrm{CL}_b(X),\ A \cap B \neq \varnothing,\ |A| \geq 2, |B| \geq 2$$
$$\text{and } \rho_1 = \rho_2 \text{ on } (A \cap B) \times (A \cap B)\}$$

Also let $\mathcal{PU}_u = \{(\rho_1, \rho_2) \in \mathcal{PU} \mid \rho_1 \text{ and } \rho_2 \text{ are uniformly continuous on } \mathrm{dom}\rho_1 \cap \mathrm{dom}\rho_2\}$.

We consider $\mathcal{PU}$ and $\mathcal{PU}_u$ as subspaces of $\mathcal{P}$. As in the case of metrics, we are able to construct an extension operator preserving ultrametrics. We obtain an analogue of Theorem 1 for ultrametrics:

Theorem 2. *There exists an operator $w\colon \mathcal{PU} \to \mathcal{UM}$ with the following properties:*

(i′) w is an extension operator;

(ii′) w is positive-homogeneous;

(iii′) for every $(\rho_1,\rho_2),(\sigma_1,\sigma_2) \in \mathcal{PU}$ with $\mathrm{dom}\rho_1 = \mathrm{dom}\sigma_1$ and $\mathrm{dom}\rho_2 = \mathrm{dom}\sigma_2$ we have $w(\max\{(\rho_1,\rho_2),(\sigma_1,\sigma_2\}) \geq \max\{w(\rho_1,\rho_2), w(\sigma_1,\sigma_2)\}$.

(iv′) the restriction $w|_{\mathcal{PU}_u} \to \mathcal{UM}$ is continuous.

Proof. We use a slight modification of the extension operator in Theorem 1 (see [9, Theorem 2.2]). Define an operator $w\colon \mathcal{PU} \to \mathcal{UM}$ by the formula

$$w(\rho_1,\rho_2)(x,y) = \inf_{a\in\mathrm{dom}\rho_1\cap\mathrm{dom}\rho_2} \max\{\rho_1(x,a),\rho_2(y,a)\}$$

for $x \in \mathrm{dom}\rho_1 \setminus \mathrm{dom}\rho_2$, $y \in \mathrm{dom}\rho_2 \setminus \mathrm{dom}\rho_1$, $(\rho_1,\rho_2) \in \mathcal{PU}$ and let $w(\rho_1,\rho_2)(x,y) = \rho_i(x,y)$ if $x,y \in \mathrm{dom}\rho_i$, $i \in \{1,2\}$. The properties of w can be checked as for the case of metrics. □

The following example shows that, in general, there is no extension operator for admissible pairs of ultrametrics which preserves maxima of ultrametrics. That is, we cannot, in general, get $w(\max\{(\rho_1,\rho_2),(\sigma_1,\sigma_2)\} = \max\{w(\rho_1,\rho_2), w(\sigma_1,\sigma_2)\}$ for every $(\rho_1,\rho_2),(\sigma_1,\sigma_2) \in \mathcal{PU}$ with $\mathrm{dom}\rho_1 = \mathrm{dom}\sigma_1$ and $\mathrm{dom}\rho_2 = \mathrm{dom}\sigma_2$.

Example 4. Suppose that there exist discrete subspaces $A = \{x,a,b\}$ and $B = \{y,a,b\}$ of X with $x \neq y$. Let ultrametrics (ρ_1,ρ_2), (ρ_1',ρ_2'), (σ_1,σ_2), $(\sigma_1',\sigma_2') \in \mathcal{PU}$ be defined as follows:

$$\begin{gathered}
\rho_1(x,a)=1,\ \rho_1(x,b)=2,\ \rho_2(y,a)=2,\ \rho_2(y,b)=2,\ \rho_1(a,b)=\rho_2(a,b)=2,\\
\rho_1'(x,a)=2,\ \rho_1'(x,b)=1,\ \rho_2'(y,a)=2,\ \rho_2'(y,b)=2,\ \rho_1'(a,b)=\rho_2'(a,b)=2,\\
\sigma_1(x,a)=1,\ \sigma_1(x,b)=2,\ \sigma_2(y,a)=1,\ \sigma_2(y,b)=2,\ \sigma_1(a,b)=\sigma_2(a,b)=2,\\
\sigma_1'(x,a)=2,\ \sigma_1'(x,b)=1,\ \sigma_2'(y,a)=2,\ \sigma_2'(y,b)=1,\ \sigma_1'(a,b)=\sigma_2'(a,b)=2.
\end{gathered}$$

Assume that there exists an operator $v\colon \mathcal{PU} \to \mathcal{UM}$ which preserves maxima of ultrametrics. Since $v(\rho_1,\rho_2)(x,a) = \rho_1(x,a) = 1$ and $v(\rho_1,\rho_2)(y,a) = \rho_2(y,a) = 2$ we should have $v(\rho_1,\rho_2)(x,y) = 2$ because $v(\rho_1,\rho_2)$ is an ultrametric. Similarly, for (ρ_1',ρ_2') we obtain $v(\rho_1',\rho_2')(x,y) = 2$.

Now the pair of ultrametrics $\max\{(\rho_1,\rho_2),(\rho_1',\rho_2')\} = (\max\{\rho_1,\rho_1'\},\max\{\rho_2,\rho_2'\})$ assigns the distance 2 to every pair of distinct points. Since v preserves maxima, we obtain $v(\max\{(\rho_1,\rho_2),(\rho_1',\rho_2')\})(x,y) = \max\{v(\rho_1,\rho_2)(x,y), v(\rho_1',\rho_2')(x,y)\} = 2$.

Now since $v(\sigma_1,\sigma_2)(x,a) = v(\sigma_1,\sigma_2)(y,a) = 1$, we see that $v(\sigma_1,\sigma_2)(x,y) \leq 1$. Similarly we obtain $v(\sigma_1',\sigma_2')(x,y) \leq 1$.

The operator v preserves maxima of ultrametrics, so we obtain

$$v(\max\{(\sigma_1,\sigma_2),(\sigma_1',\sigma_2')\})(x,y) = (\max\{v(\sigma_1,\sigma_2)(x,y), v(\sigma_1',\sigma_2')(x,y)\}) \leq 1$$

as we just noticed.

But it is clear that $\max\{(\sigma_1,\sigma_2),(\sigma_1',\sigma_2')\} = \max\{(\rho_1,\rho_2),(\rho_1',\rho_2')\}$, so we should have $v(\max\{(\sigma_1,\sigma_2),(\sigma_1',\sigma_2')\})(x,y) = 2$. A contradiction.

Our last example which is an analogue of Example 3 shows that, in general, the extension of a pair (ρ_1, ρ_2) of uniformly continuous metrics need not be uniformly continuous on $\mathrm{dom}\rho_1 \cup \mathrm{dom}\rho_2$.

Example 5. Let (C, d) be the Cantor ternary set with the ultrametric d defined as follows:

$$d(t,s) = d(\{t_n\}, \{s_n\}) = \begin{cases} \max\{2^{-k} \in \mathbb{N} \mid t_k \neq s_k\}, & \text{if } t \neq s; \\ 0, & \text{if } t = s \end{cases}$$

for every $t, s \in C$. So, we regard every point t from C as a sequence $\{t_n\} \in \{0,1\}^{\mathbb{N}}$. Let (X, d) be a subspace of $(C, d))$ defined as follows: $X = \{1/3, 1\} \cup \{3^{-2n} \mid n \in \mathbb{N}\} \cup \{3^{-2n-1} \mid n \in \mathbb{N}\}$. Consider closed subsets $A = \{1/3, 1\} \cup \{3^{-2n} \mid n \in \mathbb{N}\}$ and $B = \{1/3, 1\} \cup \{3^{-2n-1} \mid n \in \mathbb{N}\}$ of C. It is clear that $A \cap B = \{1/3, 1\}$. Let σ_1 be the metric on A which coincides with d. Construct a metric σ_2 on B so that the resulting metric space (B, σ_2) is isometric to the subspace $B' = \{1/3, 1\} \cup \{2/3 + 3^{-2n-1} \mid n \in \mathbb{N}\}$ of (C, d) where the isometry $i\colon (B, \sigma_2) \to (B', d)$ is defined as follows: $i(1/3) = 1/3$, $i(1) = 1$ and $i(3^{-2n-1}) = 2/3 + 3^{-2n-1}$ for $n \in \mathbb{N}$. One can see that (σ_1, σ_2) is an admissible pair of uniformly continuous metrics with respect to the ultrametric d on C. Let us show that there exists a unique extension $v(\sigma_1, \sigma_2)$ of the pair (σ_1, σ_2) to an ultrametric on $A \cup B$. This extension is not, however, uniformly continuous. We use denotations $a = 1/3$, $b = 1$, $x_n = 3^{-2n}$ and $y_n = 3^{-2n-1}$, $n \in \mathbb{N}$ for the elements of $A \cup B$. We need to define only the distances between x_n and y_k for $n, k \in \mathbb{N}$. We obtain

$$\begin{aligned} v(\sigma_1, \sigma_2)(x_n, y_k) &\leq \max\{v(\sigma_1, \sigma_2)(x_n, a), v(\sigma_1, \sigma_2)(a, y_k)\} = \\ &= \max\{\sigma_1(x_n, a), \sigma_2(a, y_k)\} = \max\left\{\frac{1}{4}, \frac{1}{2}\right\} = \frac{1}{2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{1}{2} = \sigma_1(x_n, b) = v(\sigma_1, \sigma_2)(x_n, b) &\leq \max\{v(\sigma_1, \sigma_2)(x_n, y_k), v(\sigma_1, \sigma_2)(y_k, b)\} = \\ &= \max\left\{v(\sigma_1, \sigma_2)(x_n, y_k), \frac{1}{4}\right\} = v(\sigma_1, \sigma_2)(x_n, y_k). \end{aligned}$$

Together these inequalities imply $v(\sigma_1, \sigma_2)(x_n, y_k) = 1/2$ for every $n, k \in \mathbb{N}$. Now $v(\sigma_1, \sigma_2)$ is not uniformly continuous on $A \cup B$ because $d(x_n, y_k)$ can be made arbitrarily close to zero by taking sufficiently large n and k, while $v(\sigma_1, \sigma_2)(x_n, y_k) = 1/2$ for all $n, k \in \mathbb{N}$.

Note that σ_1 and σ_2 are also Lipschitz ultrametrics on A and B respectively. As a consequence, we conclude that, in general, there is no extension operator preserving admissible pairs of Lipschitz ultrametrics defined on closed and bounded subsets of X.

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